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EINIGE INTEGRALDARSTELLUNGEN FÜR PRODUKTE VON WHITTAKERSCHEN FUNKTIONEN

Von C. S. MEIJER (*Groningen*)

[Received 23 March 1935]

Einleitung

In der vorliegenden Abhandlung werde ich die drei folgenden Beziehungen beweisen*

$$W_{k,m}(z^2e^{\frac{1}{2}\pi i})W_{k,m}(z^2e^{-\frac{1}{2}\pi i}) = \frac{4z^2}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\times$$
$$\times \int_0^\infty J_{-2k}(\tfrac{1}{2}v^2)K_{2m}(zve^{\frac{1}{2}\pi i})K_{2m}(zve^{-\frac{1}{2}\pi i})v\,dv, \qquad (1)$$

$$W_{k,m}(z^2e^{\frac{1}{2}\pi i})W_{k,m}(z^2e^{-\frac{1}{2}\pi i}) = -\frac{4z^2}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\times$$
$$\times \int_0^\infty K_{2k}(\tfrac{1}{2}v^2)K_{2m}(zv)\{J_{2m}(zv)\sin(m-k)\pi+Y_{2m}(zv)\cos(m-k)\pi\}v\,dv, \qquad (2)$$

$$W_{k,m}(z^2)W_{-k,m}(z^2) = -2z^2\int_0^\infty J_{2k}(\tfrac{1}{2}v^2)K_{2m}(zv)\times$$
$$\times\{J_{2m}(zv)\sin(m-k)\pi+Y_{2m}(zv)\cos(m-k)\pi\}v\,dv. \qquad (3)$$

In (1) wird $|\arg z| < \frac{1}{2}\pi$, in (3) aber $|\arg z| < \frac{1}{4}\pi$ vorausgesetzt, in (2) ist arg z beliebig. Es wird ferner angenommen

$$|R(m)|+R(k) < \tfrac{1}{2} \quad \text{in (1)},$$
$$|R(m)|+|R(k)| < \tfrac{1}{2} \quad \text{in (2)},$$
$$|R(m)|-R(k) < \tfrac{1}{2} \quad \text{in (3)}.$$

1. Spezialfälle von (1), (2) und (3)

1.1. *Integraldarstellungen für Produkte von Besselschen Funktionen.* Bekanntlich hat man†

$$K_\nu(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}}W_{0,\nu}(2\zeta); \qquad (4)$$

* $W_{k,m}(z)$ ist die Whittakersche Funktion; siehe Whittaker and Watson (**6**), chapter XVI.

Man findet noch eine andere Integraldarstellung für $W_{k,m}(z^2)W_{-k,m}(z^2)$, wenn man k durch $-k$ ersetzt in (3).

† Siehe Meijer (**2**), Formel (8).

falls $|\arg z| < \frac{1}{4}\pi$ und $|R(\nu)| < \frac{1}{2}$ ist, findet man somit, wenn man $v\sqrt{2} = u$ setzt in (3),

$$K_\nu^2(z^2) = -\pi \int_0^\infty J_0(\tfrac{1}{4}u^2)K_{2\nu}(zu)\{J_{2\nu}(zu)\sin\nu\pi + Y_{2\nu}(zu)\cos\nu\pi\}u\,du.$$

Für die Hankelschen Funktionen $H_\nu^{(1)}(\zeta)$ und $H_\nu^{(2)}(\zeta)$ gilt*

$$H_\nu^{(1)}(\zeta) = \frac{2}{\pi i}e^{-\frac{1}{2}\nu\pi i}K_\nu(\zeta e^{-\frac{1}{2}\pi i}) \tag{5}$$

und

$$H_\nu^{(2)}(\zeta) = -\frac{2}{\pi i}e^{\frac{1}{2}\nu\pi i}K_\nu(\zeta e^{\frac{1}{2}\pi i}). \tag{6}$$

Hieraus und aus (4) ergibt sich

$$H_\nu^{(1)}(\zeta)H_\nu^{(2)}(\zeta) = \frac{2}{\pi\zeta}W_{0,\nu}(2\zeta e^{\frac{1}{2}\pi i})W_{0,\nu}(2\zeta e^{-\frac{1}{2}\pi i}).$$

Ist $|\arg z| < \frac{1}{2}\pi$ und $|R(\nu)| < \frac{1}{2}$, dann hat man also, mit Rücksicht auf (1),

$$H_\nu^{(1)}(z^2)H_\nu^{(2)}(z^2) = \frac{8\cos\nu\pi}{\pi^2}\int_0^\infty J_0(\tfrac{1}{4}u^2)K_{2\nu}(zue^{\frac{1}{2}\pi i})K_{2\nu}(zue^{-\frac{1}{2}\pi i})u\,du. \tag{7}$$

Ebenso findet man mit Hilfe von (2), falls $|R(\nu)| < \frac{1}{2}$ ist,

$$H_\nu^{(1)}(z^2)H_\nu^{(2)}(z^2) = -\frac{8\cos\nu\pi}{\pi^2}\int_0^\infty K_0(\tfrac{1}{4}u^2)K_{2\nu}(zu)\times$$
$$\times\{J_{2\nu}(zu)\sin\nu\pi + Y_{2\nu}(zu)\cos\nu\pi\}u\,du.$$

Nun hat man†

$$I_\nu(w)\cos\nu\pi = \frac{1}{2\pi i}\{e^{-\nu\pi i}K_\nu(we^{-\pi i}) - e^{\nu\pi i}K_\nu(we^{\pi i})\},$$

somit wegen (5) und (6)

$$I_\nu(w)K_\nu(w)\cos\nu\pi = \frac{\pi}{8i}\{e^{-\nu\pi i}H_\nu^{(1)}(we^{-\frac{1}{2}\pi i})H_\nu^{(2)}(we^{-\frac{1}{2}\pi i}) -$$
$$-e^{\nu\pi i}H_\nu^{(1)}(we^{\frac{1}{2}\pi i})H_\nu^{(2)}(we^{\frac{1}{2}\pi i})\}.$$

Ist $|\arg z| < \frac{1}{4}\pi$ und $|R(\nu)| < \frac{1}{2}$, dann gilt also wegen (7)

$$I_\nu(z^2)K_\nu(z^2)$$
$$= \frac{1}{\pi i}\int_0^\infty J_0(\tfrac{1}{4}u^2)K_{2\nu}(zu)\{e^{-\nu\pi i}K_{2\nu}(zue^{-\frac{1}{2}\pi i}) - e^{\nu\pi i}K_{2\nu}(zue^{\frac{1}{2}\pi i})\}u\,du. \tag{8}$$

* Für (5) vergl. man Watson (5), 78 Formel (8). Weiter hat man (siehe Watson (5), 75 Formel (5) mit $m = 1$) $H_\nu^{(2)}(\zeta) = -e^{\nu\pi i}H_\nu^{(1)}(\zeta e^{\pi i})$; hieraus und aus (5) folgt (6).

† Siehe Watson (5), 80 Formeln (17) und (18).

Aus (5) und (6) folgt aber

$$\frac{1}{\pi i}\{e^{-\nu\pi i}K_{2\nu}(zue^{-\frac{1}{2}\pi i})-e^{\nu\pi i}K_{2\nu}(zue^{\frac{1}{2}\pi i})\}$$
$$=\tfrac{1}{2}\{H^{(1)}_{2\nu}(zu)+H^{(2)}_{2\nu}(zu)\}=J_{2\nu}(zu).$$

Relation (8) geht daher über in

$$I_\nu(z^2)K_\nu(z^2)=\int_0^\infty J_0(\tfrac{1}{4}u^2)K_{2\nu}(zu)J_{2\nu}(zu)u\,du. \tag{9}$$

Die rechte Seite dieser Beziehung ist konvergent für $R(\nu)>-\frac{1}{2}$. Aus der Theorie der analytischen Fortsetzung ergibt sich also, dass (9) gültig ist, falls $|\arg z|<\frac{1}{4}\pi$ und $R(\nu)>-\frac{1}{2}$ ist. Formel (9) ist neuerdings von S. C. Mitra* gegeben worden.

1.2. *Integraldarstellungen für Produkte von parabolischen Zylinderfunktionen.*

Für die parabolische Zylinderfunktion $D_n(z)$ gilt bekanntlich†

$$D_n(z)=2^{\frac{1}{2}n+\frac{1}{4}}z^{-\frac{1}{2}}W_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2).$$

Folglich hat man

$$D_n(ze^{\frac{1}{2}\pi i})D_n(ze^{-\frac{1}{2}\pi i})=2^{n+\frac{1}{2}}z^{-1}W_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2e^{\frac{1}{2}\pi i})W_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2e^{-\frac{1}{2}\pi i}) \tag{10}$$

und

$$D_n(z)D_{-n-1}(z)=z^{-1}W_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2)W_{-\frac{1}{2}n-\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2). \tag{11}$$

Nun ist‡

$$K_{-\frac{1}{2}}(\zeta)=\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}}e^{-\zeta},\qquad J_{-\frac{1}{2}}(\zeta)=\left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}}\cos\zeta,\qquad Y_{-\frac{1}{2}}(\zeta)=\left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}}\sin\zeta.$$

Aus (10) und (1) ergibt sich somit, falls $|\arg z|<\frac{1}{2}\pi$ und $R(n)<0$ ist,

$$D_n(ze^{\frac{1}{2}\pi i})D_n(ze^{-\frac{1}{2}\pi i})=\frac{\sqrt{\pi}}{\Gamma(-n)}\int_0^\infty J_{-n-\frac{1}{2}}(\tfrac{1}{2}v^2)e^{-zv}\,dv;$$

ebenso aus (10) und (2), falls $-1<R(n)<0$ ist und $v=t\sqrt{2}$ gesetzt wird,

$$D_n(ze^{\frac{1}{2}\pi i})D_n(ze^{-\frac{1}{2}\pi i})=\frac{2^{\frac{1}{2}}}{\Gamma(-n)\sqrt{\pi}}\int_0^\infty K_{n+\frac{1}{2}}(t^2)e^{-zt}\cos(zt-\tfrac{1}{2}n\pi)\,dt,$$

und schliesslich aus (11) und (3), falls $|\arg z|<\frac{1}{4}\pi$ und $R(n)>-1$ ist,

$$D_n(z)D_{-n-1}(z)=2\int_0^\infty J_{n+\frac{1}{2}}(t^2)e^{-zt}\cos(zt-\tfrac{1}{2}n\pi)\,dt.$$

* Mitra (4), 86 Formel (17).

† Whittaker and Watson (6), § 16.5.

‡ Siehe Watson (5), 80 Formel (13), 55 Formel (6), 74 Formel (4) und 54 Formel (3).

Ersetzt man hierin n durch $-n-1$, dann erhält man

$$D_n(z)D_{-n-1}(z) = -2\int_0^\infty J_{-n-\frac{1}{2}}(t^2)e^{-zt}\sin(zt+\tfrac{1}{2}n\pi)dt.$$

2. Beweis von (1)

Aus der asymptotischen Formel

$$K_\nu(w) = \left(\frac{\pi}{2w}\right)^{\frac{1}{2}}e^{-w}\left\{1+O\left(\frac{1}{w}\right)\right\} \tag{12}$$

folgt $$K_\nu(we^{\frac{1}{2}\pi i})K_\nu(we^{-\frac{1}{2}\pi i}) = \frac{\pi}{2w}e^{-w\sqrt{2}}\left\{1+O\left(\frac{1}{w}\right)\right\}.$$

Die rechte Seite von (1) ist daher für $|\arg z| < \frac{1}{2}\pi$ eine analytische Funktion von z; ich darf also $\arg z = 0$ voraussetzen.*

Nun gilt†

$$W_{k,m}(\zeta e^{\frac{1}{2}\pi i})W_{k,m}(\zeta e^{-\frac{1}{2}\pi i}) = \frac{4\zeta^{2k}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty u^{\alpha+\beta-1}K_{\alpha-\beta}(2u)\times$$

$$\times {}_4F_3\left(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k,\ \tfrac{1}{2}-k,\ 1-k;\ 1-2k,\ \alpha,\ \beta;\ -\frac{4u^2}{\zeta^2}\right)du. \tag{13}$$

Hierin sind α und β beliebige Zahlen mit $R(\alpha) > 0$, $R(\beta) > 0$. Nimmt man $\alpha = 1-k$, $\beta = \frac{1}{2}-k$ und $\zeta = z^2$ $(z > 0)$, dann erhält man‡

$$W_{k,m}(z^2e^{\frac{1}{2}\pi i})W_{k,m}(z^2e^{-\frac{1}{2}\pi i}) = \frac{2^{1-2k}z^{4k}}{\Gamma(1-2k)}\times$$

$$\times\int_0^\infty e^{-2u}u^{-2k}\,{}_2F_1\left(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k;\ 1-2k;\ -\frac{4u^2}{z^4}\right)du. \tag{14}$$

Nun hat man, falls $|R(m)|+R(k) < \frac{1}{2}$ ist§,

$$\frac{2^{1-2k}z^{4k}u^{-2k}}{\Gamma(1-2k)}\,{}_2F_1\left(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k;\ 1-2k;\ -\frac{4u^2}{z^4}\right)$$

$$= \frac{2z^2}{u\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\int_0^\infty J_{-2k}(\tfrac{1}{2}v^2)K_{2m}\left(\frac{z^2v^2}{4u}\right)v\,dv.$$

* Ich nehme an, dass $|R(m)|+R(k) < \frac{1}{2}$ ist, so dass das Integral in der rechten Seite von (1) konvergent ist.

† Siehe Meijer (**3**), Satz 6; Formel (13) gilt für $|\arg\zeta| < \frac{1}{2}\pi$.

‡ $K_{-\frac{1}{2}}(w) = (\pi/2w)^{\frac{1}{2}}e^{-w}$.

§ Siehe Watson (**5**), 410 Formel (1).

Aus (14) folgt daher*

$$W_{k,m}(z^2e^{\frac{1}{2}\pi i})W_{k,m}(z^2e^{-\frac{1}{2}\pi i}) = \frac{2z^2}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\times$$

$$\times \int_0^\infty J_{-2k}(\tfrac{1}{2}v^2)v\,dv \int_0^\infty e^{-2u}K_{2m}\left(\frac{z^2v^2}{4u}\right)\frac{du}{u}. \quad (15)$$

Es gilt aber†

$$\int_0^\infty e^{-2u}K_{2m}\left(\frac{z^2v^2}{4u}\right)\frac{du}{u} = 2K_{2m}(zve^{\frac{1}{2}\pi i})K_{2m}(zve^{-\frac{1}{2}\pi i}).$$

Formel (15) ist also äquivalent mit (1).

3. Beweis von (2)

Sind $a,\ b,\ c \neq 0, -1, -2...$, so gibt es eine von $a,\ b,\ c$ abhängige, positive Zahl τ mit der Eigenschaft

$$\left|\frac{n^3}{(a+n-1)(b+n-1)(c+n-1)}\right| < \tau^4 \qquad (n = 1, 2, 3,...).$$

Folglich hat man für beliebige komplexe Werte von w

$$\sum_{n=0}^\infty \left|\frac{\Gamma(a)\Gamma(b)\Gamma(c)w^{4n}}{n!\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}\right| \leqslant 1+\frac{(\tau|w|)^4}{(1!)^4}+\frac{(\tau|w|)^8}{(2!)^4}+...$$

$$\leqslant \left(1+\frac{\tau|w|}{1!}+\frac{(\tau|w|)^2}{2!}+...\right)^4 = e^{4\tau|w|}.$$

Hieraus und aus der asymptotischen Formel (12) folgt, dass das Integral

$$\int_0^\infty |K_\mu(\tfrac{1}{2}v^2)|\left\{\sum_{n=0}^\infty \left|\frac{\Gamma(a)\Gamma(b)\Gamma(c)\zeta^n v^{4n}}{n!\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}\right|\right\}|v^{2\lambda+1}|\,dv$$

konvergent ist.‡ Im Integral

$$\int_0^\infty K_\mu(\tfrac{1}{2}v^2)\,{}_0F_3(a,b,c;\zeta v^4)v^{2\lambda+1}\,dv$$

darf man also§ die Funktion ${}_0F_3(a,b,c;\zeta v^4)$ nach steigenden Potenzen

* Die Vertauschung der Integrationsfolge ist erlaubt; man vergl. Bromwich (1), § 177.

† Watson (5), § 13.71.

‡ Ich nehme an, dass $|R(\mu)|-R(\lambda) < 1$ ist.

§ Man vergl. Bromwich (1), 453.

von v entwickeln und gliedweise integrieren. Wegen*

$$\int_0^\infty K_\mu(\tfrac{1}{2}v^2)v^{2\alpha+1}\,dv = 2^{2\alpha-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha+\tfrac{1}{2}\mu)$$

findet man dann

$$\int_0^\infty K_\mu(\tfrac{1}{2}v^2)\,{}_0F_3(a,b,c;\zeta v^4)v^{2\lambda+1}\,dv = 2^{2\lambda-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda-\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu)\times$$
$$\times\,{}_2F_3(\tfrac{1}{2}+\tfrac{1}{2}\lambda-\tfrac{1}{2}\mu,\tfrac{1}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu;a,b,c;16\zeta). \quad (16)$$

Diese Beziehung gilt für $|R(\mu)|-R(\lambda)<1$ und alle Werte von ζ.

Nun hat man, falls ν nicht ganz ist,†

$$K_\nu(w) = \frac{\pi}{2\sin\nu\pi}\{I_{-\nu}(w)-I_\nu(w)\}, \qquad Y_\nu(w) = \frac{J_\nu(w)\cos\nu\pi - J_{-\nu}(w)}{\sin\nu\pi}.$$

Hieraus ergibt sich, falls $2m$ nicht ganz ist,

$$K_{2m}(zv)\{J_{2m}(zv)\sin(m-k)\pi+Y_{2m}(zv)\cos(m-k)\pi\}$$
$$= \frac{\pi}{2\sin^2 2m\pi}\{-I_{2m}(zv)J_{2m}(zv)\cos(m+k)\pi+I_{2m}(zv)J_{-2m}(zv)\cos(m-k)\pi+$$
$$+I_{-2m}(zv)J_{2m}(zv)\cos(m+k)\pi-I_{-2m}(zv)J_{-2m}(zv)\cos(m-k)\pi\}. \quad (17)$$

Weiter gilt‡

$$I_{2m}(zv)J_{2m}(zv) = \frac{1}{\Gamma^2(1+2m)}\left(\frac{zv}{2}\right)^{4m}{}_0F_3\left(\tfrac{1}{2}+m,1+m,1+2m;-\frac{z^4v^4}{64}\right) \quad (18)$$

und

$$I_{2m}(zv)J_{-2m}(zv) = \frac{1}{\Gamma(1+2m)\Gamma(1-2m)}\left\{{}_0F_3\left(\tfrac{1}{2},1+m,1-m;-\frac{z^4v^4}{64}\right)-\right.$$
$$\left.-\frac{z^2v^2m}{1-4m^2}\,{}_0F_3\left(\tfrac{3}{2},\tfrac{3}{2}+m,\tfrac{3}{2}-m;-\frac{z^4v^4}{64}\right)\right\}. \quad (19)$$

Aus (17), (18), (19) und (16) folgt nun, falls $|R(m)|+|R(k)|<\frac{1}{2}$ und $m\neq 0$ ist,

$$-\frac{4z^2}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\int_0^\infty K_{2k}(\tfrac{1}{2}v^2)K_{2m}(zv)\times$$
$$\times\{J_{2m}(zv)\sin(m-k)\pi+Y_{2m}(zv)\cos(m-k)\pi\}v\,dv$$

* Siehe Watson (**5**), 388 Formel (8).

† Watson (**5**), 78 Formel (6) und 64 Formel (1).

‡ Watson (**5**), 148 Formeln (3) und (4).

$$= \frac{\Gamma^2(-2m)z^{2+4m}}{\Gamma^2(\frac{1}{2}-m-k)}\,{}_2F_3(\tfrac{1}{2}+m-k, \tfrac{1}{2}+m+k; \tfrac{1}{2}+m, 1+m, 1+2m; -\tfrac{1}{4}z^4)+$$

$$+\frac{\Gamma^2(2m)z^{2-4m}}{\Gamma^2(\frac{1}{2}+m-k)}\,{}_2F_3(\tfrac{1}{2}-m-k, \tfrac{1}{2}-m+k; \tfrac{1}{2}-m, 1-m, 1-2m; -\tfrac{1}{4}z^4)+$$

$$+\frac{\Gamma(m)\Gamma(-m)z^2}{2\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\,{}_2F_3(\tfrac{1}{2}+k, \tfrac{1}{2}-k; \tfrac{1}{2}, 1+m, 1-m; -\tfrac{1}{4}z^4)+$$

$$+\frac{\Gamma(m-\frac{1}{2})\Gamma(-m-\frac{1}{2})kz^4}{2\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)}\,{}_2F_3(1+k, 1-k; \tfrac{3}{2}, \tfrac{3}{2}+m, \tfrac{3}{2}-m; -\tfrac{1}{4}z^4). \quad (20)$$

Die rechte Seite dieser Relation ist aber gleich*

$$W_{k,m}(z^2e^{\frac{1}{2}\pi i})W_{k,m}(z^2e^{-\frac{1}{2}\pi i}),$$

so dass Formel (2) bewiesen ist für den Fall, dass $m \neq 0$ ist. Durch Grenzübergang findet man, dass (2) auch noch gilt für $m = 0$.

4. Beweis von (3)

Ist $2m \neq 0, \pm 1, \pm 2, \ldots$, dann gilt bekanntlich†

$$W_{k,m}(\zeta) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}\zeta^{\frac{1}{2}+m}e^{-\frac{1}{2}\zeta}{}_1F_1(\tfrac{1}{2}+m-k; 1+2m; \zeta)+$$

$$+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\zeta^{\frac{1}{2}-m}e^{-\frac{1}{2}\zeta}{}_1F_1(\tfrac{1}{2}-m-k; 1-2m; \zeta). \quad (21)$$

Ferner hat man‡

$${}_1F_1(\alpha;\beta;-\zeta) = e^{-\zeta}{}_1F_1(\beta-\alpha;\beta;\zeta).$$

Hieraus und aus (21) geht hervor, falls $2m$ nicht ganz und

$$\tfrac{1}{2} \pm m + k \neq 0, -1, -2, \ldots \text{ ist,}$$

$$\Gamma(\tfrac{1}{2}+m+k)\Gamma(\tfrac{1}{2}-m+k)e^{\pm k\pi i}W_{-k,m}(\zeta e^{\pm\pi i})$$
$$= \Gamma(-2m)\Gamma(\tfrac{1}{2}+m+k)e^{\pm(\frac{1}{2}+m+k)\pi i}\zeta^{\frac{1}{2}+m}e^{-\frac{1}{2}\zeta}{}_1F_1(\tfrac{1}{2}+m-k; 1+2m; \zeta)+$$
$$+\Gamma(2m)\Gamma(\tfrac{1}{2}-m+k)e^{\pm(\frac{1}{2}-m+k)\pi i}\zeta^{\frac{1}{2}-m}e^{-\frac{1}{2}\zeta}{}_1F_1(\tfrac{1}{2}-m-k; 1-2m; \zeta). \quad (22)$$

Aus (22) und (21) ergibt sich nun, falls $\frac{1}{2} \pm m + k \neq 0, -1, -2, \ldots$ ist,§

$$W_{k,m}(\zeta) = \frac{\Gamma(\frac{1}{2}+m+k)\Gamma(\frac{1}{2}-m+k)}{2\pi i} \times$$
$$\times \{e^{k\pi i}W_{-k,m}(\zeta e^{\pi i}) - e^{-k\pi i}W_{-k,m}(\zeta e^{-\pi i})\}. \quad (23)$$

* Meijer (**3**), Formel (87). Die Beziehung

$$W_{k,m}(z^2e^{\frac{1}{2}\pi i})W_{k,m}(z^2e^{-\frac{1}{2}\pi i}) = \text{rechte Seite von (20)}$$

ist in (**3**) nur für $|\arg z| < \frac{1}{2}\pi$ bewiesen worden. Sie gilt aber, wegen der Theorie der analytischen Fortsetzung, für alle Werte von $\arg z$.

† Whittaker and Watson (**6**), § 16.41 und § 16.1.

‡ Watson (**5**), 102 Formel (1).

§ (23) gilt aus Stetigkeitsgründen auch noch, falls $2m$ ganz ist.

Folglich gilt

$$W_{k,m}(z^2)W_{-k,m}(z^2) = \frac{\Gamma(\frac{1}{2}+m+k)\Gamma(\frac{1}{2}-m+k)}{2\pi i}\times$$

$$\times\{e^{k\pi i}W_{-k,m}(z^2e^{\pi i})W_{-k,m}(z^2)-e^{-k\pi i}W_{-k,m}(z^2e^{-\pi i})W_{-k,m}(z^2)\}. \quad (24)$$

Ist $|\arg z| < \frac{1}{4}\pi$ und $|R(m)|-R(k) < \frac{1}{2}$, dann hat man also, mit Rücksicht auf (1),

$$W_{k,m}(z^2)W_{-k,m}(z^2) = \frac{2z^2}{\pi}\int_0^\infty J_{2k}(\tfrac{1}{2}v^2)K_{2m}(zv)\times$$

$$\times\{e^{k\pi i}K_{2m}(zve^{\frac{1}{2}\pi i})+e^{-k\pi i}K_{2m}(zve^{-\frac{1}{2}\pi i})\}v\,dv. \quad (25)$$

Aus (5) und (6) folgt aber

$$e^{k\pi i}K_{2m}(zve^{\frac{1}{2}\pi i})+e^{-k\pi i}K_{2m}(zve^{-\frac{1}{2}\pi i})$$

$$= \tfrac{1}{2}\pi i\{e^{(m-k)\pi i}H^{(1)}_{2m}(zv)-e^{(k-m)\pi i}H^{(2)}_{2m}(zv)\}$$

$$= -\pi\{J_{2m}(zv)\sin(m-k)\pi+Y_{2m}(zv)\cos(m-k)\pi\}.$$

Relation (25) ist also äquivalent mit (3).

Bemerkung. Mit Hilfe von (24) und (2) kann man noch eine andere Integraldarstellung für $W_{k,m}(z^2)W_{-k,m}(z^2)$ ableiten. Diese wird aber sehr kompliziert.

LITERATURVERZEICHNIS

1. T. J. I'A. Bromwich, *Theory of Infinite Series* (first edition).
2. C. S. Meijer, *Proc. Kon. Akad. Wet. Amsterdam*, 37 (1934), 805–12.
3. C. S. Meijer, 'Integraldarstellungen aus der Theorie der Besselschen Funktionen', *Proc. London Math. Soc.* (Diese Arbeit wird demnächst erscheinen.)
4. S. C. Mitra, *Bull. Calcutta Math. Soc.*, 25 (1933), 81–98.
5. G. N. Watson, *Theory of Bessel Functions*.
6. E. T. Whittaker and G. N. Watson, *Modern Analysis*.

ON EQUIVALENT OBSERVERS

By G. J. WHITROW (*Oxford*)

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1. Introduction

THE problem of correlating the space and time measures of two observers who are assumed to be in uniform relative motion was considered in a recent paper.* By employing a method originally suggested by Professor E. A. Milne it was shown that the Lorentz formulae could be deduced without appealing to the homogeneity of space and time. The assumptions made were

(i) the existence of a definite velocity c which is invariant for any two observers A_0, A_1 in uniform relative motion;

(ii) the existence of signals which may be transmitted with this velocity;

(iii) the relativity postulate: that the formulae connecting the space and time measures of A_0 with those of A_1 are reversible.

Further, it was assumed that, if the velocity of A_1 according to A_0 is V, then the velocity of A_0 according to A_1 is $-V$, and it was pointed out, in a footnote, that this could be regarded as a consequence of the assumption that A_0 and A_1 have equivalent clocks.

In his standard treatise Eddington† remarks that one rule by which we can extend fictitious time-partitions throughout the world is to assume that, if a clock be moved with infinitesimal velocity from one place to another, then it will continue to read the same time at its new station. Since the ideas involved in this rule are not precise, no use was made of it in the previous paper. Instead, the analysis was based on a set of operations which, in principle, could actually be carried out. In a series of lectures delivered at Oxford in October 1933, Milne showed how the procedure adopted for observers in uniform relative motion could be extended to observers in any relative motion. On a new definition of the concept of equivalence he erected an analysis which included the one-dimensional Lorentz analysis as a particular case.

In this note an analysis which is formally equivalent to Milne's

* Whitrow, *Quart. J. of Math.* (Oxford), 4 (1933), 161–72.

† Eddington, *The Mathematical Theory of Relativity* (1930), 29.

will be developed from a different point of view. In particular, the problem of correlating the observations of a linear system of equivalent observers will lead to a set of functional equations identical with those obtained by Milne.* The mathematical core of this paper will consist in the consideration of these equations. It will subsequently be shown that it is formally possible for each member to regraduate his clock so that they all appear to be in uniform motion relative to each other. Consequently, uniform motion may be described without introducing any indefinable concepts.

2. Two equivalent observers

We begin by considering two observers A_0 and A_1, each of whom is provided with some definite means of recording the order of events at himself. A signal, e.g. a light pulse, emitted by A_0 at time t_0 is received by A_1 at time t_1. By repeating this process sufficiently often it is possible in principle to determine the functional relationship of t_1 to t_0. Let this be expressed in the form

$$t_1 = \theta_{01}(t_0). \tag{1}$$

Similarly, if at time t_1', say, A_1 emits a signal which is received by A_0 at time t_0', a functional relation

$$t_0' = \theta_{10}(t_1'), \tag{2}$$

say, may be determined. *If the functional identity*

$$\theta_{10} \equiv \theta_{01}$$

holds, then A_0 and A_1 will be said to be equivalent observers.

This appears to be a different definition of equivalence from that adopted by Milne,† and in order to compare the two we suppose the signal which leaves A_0 at t_0 to be instantaneously reflected on arrival at A_1. Hence equation (2) may be rewritten in the form

$$t_0' = \theta_{10}(t_1). \tag{3}$$

The epoch of the arrival of the signal at A_1 is t_1 according to A_1. The first question Milne asks is: What will be its epoch according to A_0? As it is assumed that observers can only record local time-

* Since this paper was written Milne has published his *Relativity, Gravitation, and World-Structure* (Oxford, 1935), and references to this work, quoted as W.S., have been inserted. In the present connexion see W.S. §§ 39–45. I should like to take this opportunity of expressing my deep gratitude to Professor Milne for his constant encouragement and advice.

† Milne, W.S. § 17.

observations, the answer is purely conventional. Milne defines the epoch according to A_0 of the arrival of the signal at A_1 as

$$t = \tfrac{1}{2}(t_0' + t_0).$$

Since t_0 and t_0' are related to t_1 by equations (1) and (3), it follows that if

$$t_1 = f_{01}(t)$$

there must exist an operational identity

$$f_{01}^{-1} \equiv \tfrac{1}{2}(\theta_{10} + \theta_{01}^{-1}). \tag{4}$$

Milne also stipulates that A_0 defines the distance of A_1 at t as $c\phi_{01}(t)$ where

$$\phi_{01}(t) = \tfrac{1}{2}(t_0' - t_0) \tag{5}$$

and c is an arbitrary constant. Substituting from equations (1), (3), (4) in equation (5) we deduce the operational identity

$$\phi_{01} \equiv \tfrac{1}{2}(\theta_{10} f_{01}^{-1} - \theta_{01}^{-1} f_{01}^{-1}). \tag{6}$$

A_1 defines the epoch and distance of events at A_0 in a similar manner and deduces operational identities of the form

$$f_{10}^{-1} \equiv \tfrac{1}{2}(\theta_{01} + \theta_{10}^{-1}), \qquad \phi_{10} \equiv \tfrac{1}{2}(\theta_{01} f_{10}^{-1} - \theta_{10}^{-1} f_{10}^{-1}).$$

Consequently, if $\theta_{10} \equiv \theta_{01}$, then $f_{10} \equiv f_{01}$ and $\phi_{10} \equiv \phi_{01}$. Conversely, it is easily proved that, if $f_{10} \equiv f_{01}$ and $\phi_{10} \equiv \phi_{01}$, then $\theta_{10} \equiv \theta_{01}$. Since the functions θ correspond to Milne's auxiliary functions p, it is clear that the definition of equivalence adopted here is formally identical with that employed by Milne. In his work, however, the functions f and ϕ are of primary importance. In the present work emphasis is laid on θ, i.e. the function expressing the time of reception of a signal by one observer in terms of its time of emission from the other, both times being, of course, local times. As it is convenient to have a name for this function it will be called *the signal-function*.

3. Linear systems of equivalent observers

Let A_0, A_1, A_2 be any three observers moving in alinement with each other. If A_0, A_1 be equivalent and if A_0, A_2 be equivalent, it is not necessarily the case that A_1, A_2 should be equivalent. Milne has already discovered the necessary and sufficient conditions that this should be so, and the following is a concise alternative derivation of these conditions.

We will assume that A_0 lies between A_1 and A_2. There is no difficulty in rearranging the following argument when this is not so.

A_1 at time t_1 emits a signal which passes A_0 at t_0 and arrives at

A_2 at t_2. This process may be repeated arbitrarily often and the following functional relations, say, be determined:

$$t_0 = \theta_{10}(t_1), \qquad t_2 = \theta_{02}(t_0).$$

Moreover, let the relation between t_2 and t_1 be written

$$t_2 = \theta_{12}(t_1).$$

Then for consistency, $\qquad \theta_{12} \equiv \theta_{02}\theta_{10}.$

Similarly, at time t_2', say, A_2 emits a signal which passes A_0 at t_0' and arrives at A_1 at t_1'. This process may be repeated and the following relations be determined:

$$t_0' = \theta_{20}(t_2'), \qquad t_1' = \theta_{01}(t_0').$$

If the relation between t_1' and t_2' be written

$$t_1' = \theta_{21}(t_2'),$$

it follows that $\qquad \theta_{21} \equiv \theta_{01}\theta_{20}.$

By definition, if A_0 and A_1 are equivalent, $\theta_{10} \equiv \theta_{01}$. Similarly, if A_0 and A_2 are equivalent, $\theta_{20} \equiv \theta_{02}$. Hence

$$\theta_{21} \equiv \theta_{10}\theta_{02}.$$

If A_1 and A_2 are equivalent, then $\theta_{21} \equiv \theta_{12}$. This is so, if and only if

$$\theta_{10}\theta_{02} \equiv \theta_{02}\theta_{10}.$$

In general, if A_p, A_q, A_r be any three members of a linear system of equivalent observers taken in order, and if the signal-function of A_p, A_q be θ_{pq} and so on, we must have

$$\theta_{pr} \equiv \theta_{pq}\theta_{qr} \equiv \theta_{qr}\theta_{pq}. \tag{7}$$

These are the conditions obtained by Milne.* We notice that any two signal-functions with a common suffix are commutative.†

4. Commutative functional operators

In the previous section we have been led to consider functions θ, θ_0, say, of a single variable satisfying operational identities of the form

$$\theta\theta_0 \equiv \theta_0\theta. \tag{8}$$

In investigating identities of this nature it is convenient to express the operators in a certain canonical form analogous to a well-known

* Milne, W.S. § 43.

† The operational product of any two signal functions without a common suffix has no physical significance.

form in the theory of matrices. Connected with this representation is a famous functional equation, first investigated by Abel.*

It is easily seen that the operators commutative with a given operator form a group. For if θ_1, θ_2 be any two operators which are commutative with θ_0, then

$$\theta_1\theta_2\theta_0 \equiv \theta_1\theta_0\theta_2 \equiv \theta_0\theta_1\theta_2,$$

and so the operational product $\theta_1\theta_2$ is commutative with θ_0. The other conditions for a group are also satisfied.†

Since θ_0 is itself a member of the group, then so is $\theta_0\theta_0$, which we write as θ_0^2. More generally θ_0^n, where n is any positive integer, is a member of the group. It is evident that, if a meaning can be attached to the symbol θ_0^n when n is any real number, then the group of operators θ satisfying (8) must be continuous. Assuming provisionally that this is so, it immediately follows that the group must contain an infinitesimal member.‡ Let the function obtained by operating with this infinitesimal member on x be

$$x+\epsilon\omega(x),$$

where powers of ϵ higher than the first may be neglected. Since this infinitesimal member is commutative with θ_0, we have the identity

$$\theta_0\{x+\epsilon\omega(x)\} \equiv \theta_0(x)+\epsilon\omega\theta_0(x).$$

Assuming, provisionally, that the function on the left may be expanded by the mean-value theorem, we find on equating coefficients of ϵ that

$$\frac{d\theta_0(x)}{\omega\theta_0(x)} = \frac{dx}{\omega(x)}.$$

Writing

$$\Omega(x) \equiv \int^x \frac{dx}{\omega(x)} \tag{9}$$

then

$$\Omega\theta_0(x) = \Omega(x)+\lambda,$$

where λ is an arbitrary constant of integration. It follows from this equation that $\theta_0(x)$ may be expressed in the form

$$\theta_0(x) = \Omega^{-1}\{\Omega(x)+\lambda\}. \tag{10}$$

If we write

$$\Omega \equiv \log\psi^{-1} \tag{11}$$

* Abel, *Œuvres*, 2 (1881), 36; Schröder, *Math. Ann.* 3 (1870), 296; Picard, *Leçons sur quelques équations fonctionnelles* (1928), ch. IV.

† Bôcher, *Introduction to Higher Algebra* (1931), 82.

‡ Ince, *Ordinary Differential Equations* (1927), 95.

we may express θ_0 in the form

$$\theta_0 = \psi\alpha\psi^{-1} \tag{12}$$

where $\alpha = \exp\lambda$.

This suggests that we should consider whether it is possible to express any functional operator in the canonical form $\psi\alpha\psi^{-1}$.* It is immediately seen that, if θ be expressible in this form for any given positive α ($\neq 1$), then it is expressible in a similar form for any other α ($\neq 1$). This follows from the fact that, if

$$\phi(x) \equiv \psi(x^{1/r}),$$

then

$$\phi\alpha^r\phi^{-1} \equiv \psi\alpha\psi^{-1}.$$

By a suitable choice of r we can make α^r equal to any given positive number, other than unity.

If ψ be such that

$$\psi\alpha\psi^{-1} \equiv \theta, \tag{13}$$

then, for any argument z,

$$\psi(\alpha z) \equiv \theta\psi(z). \tag{14}$$

This functional equation may be transformed into a first-order difference equation by a method due to Laplace.†

Let

$$\left.\begin{aligned} z &= u(t), & \alpha z &= u(t+1) \\ \psi(z) &= v(t), & \psi(\alpha z) &= v(t+1) \end{aligned}\right\}. \tag{15}$$

Then

$$u(t+1) = \alpha u(t) \tag{16}$$

and

$$v(t+1) = \theta v(t), \tag{17}$$

while ψ is related to u and v by the identity

$$v(t) \equiv \psi u(t). \tag{18}$$

A solution of (16) is

$$u(t) = \alpha^t.$$

It thus remains to find a solution of (17), for then a solution for ψ is given by

$$\psi(\tau) \equiv v(\log\tau/\log\alpha).$$

Let a function v be defined thus. If $0 \leqslant x \leqslant 1$, let $v(x)$ be arbitrary save for the boundary condition

$$v(1) = \theta v(0).$$

Let

$$v(1+x) = \theta v(x)$$

and generally let

$$v(n+x) = \theta^n v(x),$$

where θ^n denotes the operational product $\theta\theta\ldots\theta$ taken n times if n is

* It is convenient to take $\alpha > 0$, so that $\lambda = \log\alpha$ is always real.

† Boole, *Calculus of Finite Differences* (1880), 302.

a positive integer, or the operational product $\theta^{-1}\theta^{-1}...\theta^{-1}$ taken $-n$ times if n is a negative integer. Then

$$v(n+1+x) = \theta v(n+x)$$

for all integers n and all numbers x in $(0, 1)$. Hence a solution, and in point of fact the general solution of (17), may be expressed in the form

$$v(t) = \theta^n v(x),$$

where

$$n = [t]^*$$

and

$$x = t - [t],$$

while, for

$$0 \leqslant x < 1,$$

$v(x)$ is an arbitrary function. If θ be a continuous function, the supplementary condition

$$v(1) = \theta v(0)$$

ensures that v, as defined above, is also continuous.† *Hence any functional operator θ may be expressed in the canonical form $\psi\alpha\psi^{-1}$.*

We now proceed to determine the general form of any functional operator θ which is commutative with a given θ_0. Writing

$$\theta_0 \equiv \psi\alpha_0\psi^{-1} \tag{19}$$

let us define an operator f by the relation

$$f \equiv \psi^{-1}\theta\psi,$$

whence

$$\theta \equiv \psi f \psi^{-1}. \tag{20}$$

Then, if

$$\theta\theta_0 \equiv \theta_0\theta,$$

the operator f satisfies the identity

$$f(\alpha_0 x) \equiv \alpha_0 f(x). \tag{21}$$

Consequently the general form of f is given by

$$f(x) \equiv xg(x),$$

where

$$g(\alpha_0 x) \equiv g(x).$$

Hence

$$g(x) \equiv h(\log x),$$

where h is an arbitrary function of period $\log \alpha_0$. We thus see that the general form of any functional operator θ which is commutative with

$$\theta_0 \equiv \psi\alpha_0\psi^{-1}$$

is given by

$$\theta \equiv \psi f \psi^{-1},$$

* $[t]$ denotes the greatest integer in t.

† Further obvious boundary conditions will ensure that v has continuous derivatives provided that θ has.

where f is an operator satisfying the identity

$$f(x) \equiv xg(\log x)$$

and g is any function of period $\log \alpha_0$.

We may now obtain the general solution of the identities

$$\theta_{pr} \equiv \theta_{pq}\theta_{qr} \equiv \theta_{qr}\theta_{pq},$$

for it follows that, if any particular θ_{pq} be expressed as $\psi\alpha_0\psi^{-1}$, then all the operators θ_{qr} are included in the form $\psi f\psi^{-1}$ where $f(x)/x$ may be expressed as a function of $\log x$ of period $\log \alpha$. In particular, $\psi\alpha\psi^{-1}$ where α is any constant is a θ_{qr}. Consequently all the operators $\psi f\psi^{-1}$ must be commutative with $\psi\alpha\psi^{-1}$. Hence $f(x)/x$ must be a function of $\log x$ admitting every real number as a period and so must be a constant. Thus the operators corresponding to the signal-functions correlating the respective members of a linear system of equivalent observers are expressible in the canonical form $\psi\alpha\psi^{-1}$ where ψ is an operator defining the system and α is a number defining a particular pair of the system.

5. Regraduation of the time-scale

Consider any two members A_p, A_q of a linear system of equivalent observers. A signal emitted by A_p at time t_p is received by A_q at time t_q. Let the functional relationship between these time readings be given by

$$t_q = \theta_{pq}(t_p), \tag{22}$$

where

$$\theta_{pq} \equiv \psi\alpha_{pq}\psi^{-1}. \tag{23}$$

Now let A_p and A_q each regraduate their time-scales in an identical manner so that the new time-readings τ_p, τ_q corresponding to t_p, t_q respectively be related to them by the formula

$$t = \psi\phi^{-1}(\tau). \tag{24}$$

Then it follows from (22) that

$$\tau_q = \vartheta_{pq}(\tau_p), \tag{25}$$

where

$$\vartheta_{pq} \equiv \phi\psi^{-1}\theta_{pq}\psi\phi^{-1}. \tag{26}$$

Substituting for θ_{pq} it immediately follows that

$$\vartheta_{pq} \equiv \phi\alpha_{pq}\phi^{-1}. \tag{27}$$

Consequently, if each member of a linear system of equivalent observers regraduates his clock in the same manner the members remain equivalent, as we should expect. The operators ψ, ϕ may be regarded as specifying the different time-scales employed.

It follows from (4) and (6) that, if the signal-function of two equivalent observers is a constant multiple α of the time, then they are in uniform relative motion of velocity V where

$$\alpha = \left(\frac{1+V/c}{1-V/c}\right)^{\frac{1}{2}}. \tag{28}$$

Hence, if A_p, A_q,... transform from the scale t to the scale τ by the formula

$$t = \psi(\tau), \tag{29}$$

the system reduces to a set of observers in uniform relative motion. Moreover, since any operator θ can be expressed in the form $\psi\alpha\psi^{-1}$, it follows that any pair of equivalent observers can choose their time-scales so that they appear to each other to be in uniform motion. *Consequently uniform relative motion may be described without introducing any indefinable concepts.*

6. The Doppler effect

It is well known that, if a source of radiation be in motion with respect to an observer, its spectral lines are displaced. This follows from the fact that the time-interval between successive wave-crests of the radiation as measured by an observer at rest with respect to the source is not equal to the time-interval between their reception.* We proceed to investigate this effect by correlating the rates of emission and reception of signals by equivalent observers.†

Let A_0 emit signals at a rate $\nu_0(t_0)$ per unit time by his clock t_0. Let A_1 receive these signals at a rate $\nu_1(t_1)$ per unit time by his clock t_1. *Then the ratio*

$$D(t) = \frac{\nu_0(t)}{\nu_1(t)} = \frac{\textit{rate of emission (proper frequency)}}{\textit{rate of reception (apparent frequency)}}$$

will be called the Doppler ratio.

A signal emitted by A_0 at t_0 is received by A_1 at t_1, say. Let a signal emitted by A_0 at $t_0+\delta t_0$ be received by A_1 at $t_1+\delta t_1$. Since the number of signals emitted by A_0 in the interval δt_0 must be equal to the number of signals received by A_1 in the interval δt_1, it follows that

$$\nu_0(t_0)\,\delta t_0 = \nu_1(t_1)\,\delta t_1.$$

* McCrea, *Relativity Physics* (1935), 35–6.

† I wish to record my deep indebtedness to Dr. A. G. Walker in connexion with the ideas of this section.

Also, since $$t_1 = \theta_{01}(t_0),$$

then $$\delta t_1 = \theta'_{01}(t_0)\,\delta t_0 = \theta'_{01}\{\theta_{01}^{-1}(t_1)\}\,\delta t_0.$$

Hence $$\nu_1(t_1)\,\delta t_1 = \frac{\nu_0\{\theta_{01}^{-1}(t_1)\}\,\delta t_1}{\theta'_{01}\{\theta_{01}^{-1}(t_1)\}},$$

and so the Doppler ratio observed by A_1 is

$$D(t_1) = \frac{\nu_0(t_1)}{\nu_1(t_1)} = \frac{\nu_0(t_1)}{\nu_0\{\theta_{01}^{-1}(t_1)\}}\,\theta'_{01}\{\theta_{01}^{-1}(t_1)\}. \tag{30}$$

If A_0 emits signals uniformly so that ν_0 is constant, this formula reduces to the simpler form

$$D(t_1) = \theta'_{01}\{\theta_{01}^{-1}(t_1)\}. \tag{31}$$

This is the formula given by Milne.* In particular, if

$$\theta_{01}(t) \equiv \alpha t,$$

then $$D(t_1) = \alpha.$$

It thus follows that, if two equivalent observers have a uniform radial velocity of recession V, then the Doppler shift observed by either of signals emitted uniformly by the other is given by

$$\frac{\nu}{\nu'} = \left(\frac{1+V/c}{1-V/c}\right)^{\frac{1}{2}}, \tag{32}$$

where ν is the proper frequency and ν' is the apparent frequency. This is Einstein's formula for the Doppler effect in the special theory of relativity.

We now consider the effect of A_0 and A_1 each changing their time-scales from t to τ where $$t = \psi(\tau).$$

The frequency-functions of emission and reception will now be denoted by $\tilde{\nu}_0(t_0)$ and $\tilde{\nu}_1(t_1)$ respectively and the intervals $(\tau_0, \tau_0+\delta\tau_0)$, $(\tau_1, \tau_1+\delta\tau_1)$ will correspond to the intervals $(t_0, t_0+\delta t_0)$, $(t_1, t_1+\delta t_1)$ respectively. By considering the number of signals emitted in the interval δt_0 we deduce the relation

$$\tilde{\nu}_0(\tau_0)\,\delta\tau_0 = \nu_0(t_0)\,\delta t_0.$$

Hence $$\tilde{\nu}_0(\tau_0) \equiv \nu_0\{\psi(\tau_0)\}\,\psi'(\tau_0).$$

If the new signal-function be ϑ_{01}, it is given in terms of the original signal-function by the formula

$$\vartheta_{01} \equiv \psi^{-1}\theta_{01}\psi.$$

* Milne, W.S. § 21.

If the new Doppler ratio be Δ, we have, by analogy with the previous work,

$$\Delta(\tau_1) = \frac{\bar{\nu}_0(\tau_1)}{\bar{\nu}_0 \vartheta_{01}^{-1}(\tau_1)} \vartheta'_{01}\{\vartheta_{01}^{-1}(\tau_1)\}$$

$$= \frac{\nu_0 \psi(\tau_1)}{\nu_0 \theta_{01}^{-1} \psi(\tau_1)} \frac{\psi'(\tau_1)}{\psi'\{\psi^{-1}\theta_{01}^{-1}\psi(\tau_1)\}} \{\psi^{-1}\theta_{01}\psi\}'\psi^{-1}\theta_{01}^{-1}\psi(\tau_1).$$

Since $$\{\psi^{-1}\theta_{01}\psi\}'\tau \equiv \frac{1}{\psi'\{\psi^{-1}\theta_{01}\psi(\tau)\}} \theta'_{01}\psi(\tau)\psi'(\tau),$$

this formula reduces to

$$\Delta(\tau_1) = \frac{\nu_0 \psi(\tau_1)}{\nu_0 \theta_{01}^{-1}\psi(\tau_1)} \theta'_{01}\{\theta_{01}^{-1}\psi(\tau_1)\}$$

$$= \frac{\nu_0(t_1)}{\nu_0 \theta_{01}^{-1}(t_1)} \theta'_{01}\{\theta_{01}^{-1}(t_1)\},$$

and so we see that $$\Delta(\tau_1) = D(t_1). \tag{33}$$

Hence the numerical value of the Doppler ratio at any instant is independent of the time-scale. In particular, if D is constant, then so is Δ.

It follows that our general theory of the Doppler effect is consistent with physical observation, for we know that the spectral shift of a moving light-source may be regarded as a colour shift. Consequently it must be independent of the time-scale.

7. A one-dimensional universe of discrete particles

In conclusion, we will apply the theory of signal-functions to the construction of a one-dimensional universe of discrete particles satisfying the cosmological principle.* This universe is described in the same way by each observer associated with a particle. If the particles be arranged in a definite order, so that A_{r+1} is on the same side of A_r as A_r is of A_{r-1}, it follows, by hypothesis, that the relations of each pair must be the same. Consequently the signal-functions connecting each pair of consecutive observers must be the same. Hence, if θ_{pq} be the functional operator connecting A_p, A_q, we must have

$$\theta_{r-1,r} \equiv \theta_{01}. \tag{34}$$

Since $$\theta_{0r} \equiv \theta_{01}\theta_{12}\theta_{23}\ldots\theta_{r-1,r},$$

it follows that $$\theta_{0r} \equiv \theta_{01}^r,$$

* Milne, W.S. §§ 46–7.

and, since $$\theta_{rs} \equiv \theta_{0r}^{-1}\theta_{0s},$$

we see that $$\theta_{rs} \equiv \theta_{01}^{s-r}. \tag{35}$$

In the particular case in which A_0 and A_1 are in uniform relative motion, so that $$\theta_{01} = \alpha, \tag{36}$$

all the observers will be in uniform relative motion and

$$\theta_{0n} = \alpha^n.$$

If V_{01} denote the relative velocity of A_0, A_1, and if V_{0n} denote the relative velocity of A_0, A_n, then

$$\frac{1+V_{0n}/c}{1-V_{0n}/c} = \left(\frac{1+V_{01}/c}{1-V_{01}/c}\right)^n. \tag{37}$$

This is the equation obtained by Milne.*

8. Summary

In his book *Relativity, Gravitation, and World-Structure* Milne has shown how equivalent observers may correlate the space and time coordinates, which they assign to events, on the sole basis of their awareness of a temporal sequence. The analysis is developed in terms of two functions correlating the epochs and distances assigned to each other by two equivalent observers. An auxiliary function p is introduced to facilitate the analysis.

It is here shown that the entire analysis may be developed in an extremely simple fashion by starting with this auxiliary function, which, for physical reasons, is called the signal-function.

The operators corresponding to the signal-functions correlating the respective members of a linear system of equivalent observers are commutative with one another. They are therefore shown to be expressible in the canonical form $\psi\alpha\psi^{-1}$ where ψ is an operator defining the system and α is a number defining a particular pair of the system.

It is shown that it is formally possible for each member to regraduate his clock so that they all appear to be in uniform motion relative to each other. Consequently uniform motion may be described without introducing any indefinable concepts.

A general theory of the Doppler effect is developed and it is also shown that signal-functions may be used to construct a one-dimensional universe of discrete particles satisfying the cosmological principle.

* Loc. cit. See also Milne, *Quart. J. of Math.* (Oxford), 5 (1934), 30–3.

ON TARRY'S PROBLEM (I)

By E. MAITLAND WRIGHT (*Oxford*)

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1. In this paper all small italic letters denote positive integers. We define $N(k)$ as the least value of j such that the equations

$$a_1^h+\ldots+a_j^h = b_1^h+\ldots+b_j^h \qquad (1 \leqslant h \leqslant k) \qquad (1)$$

have a non-trivial solution, i.e. a solution in which the b are not a permutation of the a. We define $M(k)$ as the least value of j such that (1) has a solution with

$$a_1^{k+1}+\ldots+a_j^{k+1} \neq b_1^{k+1}+\ldots+b_j^{k+1}.$$

Such a solution is necessarily non-trivial, so that

$$N(k) \leqslant M(k).$$

The value of $N(k)$ for particular values of k has been studied by a large number of writers. A whole chapter of Dickson's *History* (Dickson, **3**) is devoted to an account of their results. Until very recently the only result proved about the order of $N(k)$ was that due to Tarry (**5**), namely,

$$N(k) \leqslant 2^{k-1}.$$

Last year Pillai (**4**) proved that

$$N(k) = O(k^{-\frac{1}{2}}2^k).$$

At about the same date I described (Wright, **7**) a method (i) of finding upper bounds for $N(k)$ for successive values of k by numerical calculations, and (ii) of deducing an order-result for $N(k)$ for general k from any upper bound for $N(k)$ for a particular k. Thus I found that $N(18) \leqslant 80$ and deduced that $N(k) = O\{(160)^{k/19}\}$. Chowla (**1**) recently improved my numerical calculations to prove that $N(18) \leqslant 68$ and $N(k) = O\{(136)^{k/19}\}$. These methods cannot, however, prove anything better than $N(k) = O(e^{\alpha k})$, where α is a constant whose value can be reduced by improvements of the numerical calculations. These methods prove the same order-results for $M(k)$.*

I shall prove here

THEOREM 1.
$$N(k) \leqslant \tfrac{1}{2}(k^2+3) \qquad (k \text{ odd}),$$
$$N(k) \leqslant \tfrac{1}{2}(k^2+4) \qquad (k \text{ even}).$$

* The $M(k)$ of this paper is the same as the $J(k+2)$ of my former papers (Wright, **7**, **8**). In these I used $M(k)$ for applications for which $N(k)$ is useless.

The proof is very simple. In fact, if we confined ourselves to proving the result

$$N(k) \leqslant \tfrac{1}{2}(k^2+k+2),$$

the proof would occupy about eight lines. Unfortunately, the method does not apply to $M(k)$ and the greater part of this paper is devoted to the proof of

THEOREM 2. $$M(k) \leqslant N(k^2).$$

From this and Theorem 1 we at once deduce

THEOREM 3. $$M(k) \leqslant \tfrac{1}{2}(k^4+3) \qquad (k \text{ odd}),$$
$$M(k) \leqslant \tfrac{1}{2}(k^4+4) \qquad (k \text{ even}).$$

It is known that $M(k) \geqslant N(k) \geqslant k+1$, and it has been proved that $M(k) = N(k) = k+1$ for all $k \leqslant 7$. My own conjecture as to the real truth is that $N(k) = k+1$. If this is so, then $M(k) = N(k)$, since otherwise $N(k+1) = k+1$, and this is impossible.

2. *Proof of Theorem* 1. We take $m \geqslant 1$, $t \geqslant 2$, $n \geqslant t!\,t^m$, and $1 \leqslant a_r \leqslant n$ $(1 \leqslant r \leqslant t)$. Then there are n^t different sets $a_1,\ldots, a_t$. Since there are at most $t!$ permutations of t objects, there are not less than

$$(t!)^{-1}n^t \geqslant t^m n^{t-1}$$

sets $a_1,\ldots, a_t$, none of which is a permutation of any other. We write

$$l_h = a_1^h+\ldots+a_t^h.$$

Then $$t \leqslant l_h \leqslant tn^h.$$

If $k = 1$, Theorem 1 is trivial. Let k be odd $(k \geqslant 3)$ and take $m = \frac{1}{2}(k-1)$, $t = m(m+1)+1$. Then the number of different sets $l_2, l_4,\ldots, l_{2m}$ is at most

$$\prod_{h=1}^{m}(tn^{2h}-t+1) < t^m \prod_{h=1}^{m} n^{2h} = t^m n^{t-1},$$

and so there are at least two sets $a_1,\ldots, a_t$, not permutations of one another, which have the same $l_2, l_4,\ldots, l_{2m}$; that is, there is a non-trivial solution of

$$a_1^{2h}+\ldots+a_t^{2h} = b_1^{2h}+\ldots+b_t^{2h} \qquad (1 \leqslant h \leqslant m).$$

It follows that

$$\sum_{r=1}^{t}\{(n+1+a_r)^h+(n+1-a_r)^h\} = \sum_{r=1}^{t}\{(n+1+b_r)^h+(n+1-b_r)^h\}$$

for $1 \leqslant h \leqslant 2m+1 = k$. This is a solution of (1) of the type required with

$$j = 2t = 2m(m+1)+2 = \tfrac{1}{2}(k^2+3).$$

Hence, if k is odd, $N(k) \leqslant \frac{1}{2}(k^2+3)$.

If k is even, take $m = \frac{1}{2}k$, $t = m^2+1$. The number of different sets $l_1, l_3, \ldots, l_{2m-1}$ is at most

$$\prod_{h=1}^{m}(tn^{2h-1}-t+1) < t^m \prod_{h=1}^{m} n^{2h-1} = t^m n^{t-1},$$

and so there is a non-trivial solution of

$$a_1^{2h-1}+\ldots+a_t^{2h-1} = b_1^{2h-1}+\ldots+b_t^{2h-1} \qquad (1 \leqslant h \leqslant m).$$

Hence

$$\sum_{r=1}^{t}\{(n+1+a_r)^h+(n+1-b_r)^h\} = \sum_{r=1}^{t}\{(n+1-a_r)^h+(n+1+b_r)^h\}$$

for $1 \leqslant h \leqslant 2m = k$, and so

$$N(k) = 2t = 2m^2+2 = \tfrac{1}{2}(k^2+4).$$

3. *Proof of Theorem* 2. We now remove the restriction $1 \leqslant a_r \leqslant n$. We write

$$l_h = a_1^h+\ldots+a_j^h,$$

and use the notation

$$\{h_1, h_2, \ldots, h_r\} \to \{H_1, H_2, \ldots, H_s\} \tag{2}$$

to denote that, for the particular value of j we are considering, all sets $a_1, \ldots, a_j$ which have the same $l_{h_1}, l_{h_2}, \ldots, l_{h_r}$ have also the same $l_{H_1}, l_{H_2}, \ldots, l_{H_s}$. If this is true, l_{H_1}, for example, may be regarded as a one-valued function of $l_{h_1}, \ldots, l_{h_r}$ defined only when the equations

$$l_h = a_1^h+\ldots+a_j^h \qquad (h = h_1, h_2, \ldots, h_r)$$

are soluble in positive integers; l_{H_1} is undefined for all other sets of values of $l_{h_1}, \ldots, l_{h_r}$. We use

$$\{h_1, h_2, \ldots, h_r\} \nrightarrow \{H_1, \ldots, H_s\}$$

to denote that (2) is not true.

LEMMA 1. *If* $\{1, 2, \ldots, k\} \nrightarrow k+1$, *then* $j \geqslant M(k)$.

This is simply a restatement of the definition of $M(k)$.

LEMMA 2. *If* $\{1, 2, \ldots, k\} \to \{k+1, k+2, \ldots, j\}$, *then* $j < N(k)$.

If $l_1, \ldots, l_j$ are all one-valued functions of $l_1, \ldots, l_k$, then so are the coefficients in the equation

$$x^j+p_1x^{j-1}+\ldots+p_j = 0,$$

whose roots are $a_1,\ldots, a_j$. Hence, apart from permutations, there is at most one set $a_1,\ldots, a_j$ corresponding to each set $l_1,\ldots, l_k$.

LEMMA 3. *If* $\{1, 2,\ldots, k\} \to k+1$, *then*

$$\{r, 2r,\ldots, kr\} \to (k+1)r.$$

Let us suppose that

$$\{r, 2r,\ldots, kr\} \nrightarrow (k+1)r$$

Then there exist two sets $a_1,\ldots, a_j$ and $b_1,\ldots, b_j$, such that

$$a_1^{rh}+\ldots+a_j^{rh} = b_1^{rh}+\ldots+b_j^{rh} \qquad (1 \leqslant h \leqslant k),$$

$$a_1^{r(k+1)}+\ldots+a_j^{r(k+1)} \neq b_1^{r(k+1)}+\ldots+b_j^{r(k+1)}.$$

Hence $a_1^r,\ldots, a_j^r$ and $b_1^r,\ldots, b_j^r$ are two sets with the same $l_1,\ldots, l_k$ and different l_{k+1}; that is,

$$\{1, 2,\ldots, k\} \nrightarrow k+1.$$

LEMMA 4. *If* $t \geqslant 2$ *and if* $\{1, 2,\ldots, k\} \to k+t$, *then*

$$\{1, 2,\ldots, k\} \to \{k+1, k+2,\ldots, k+t\}.$$

We take d any positive integer and write

$$\begin{aligned} l'_h &= (a_1+d)^h+\ldots+(a_j+d)^h \\ &= l_h+\binom{h}{1}dl_{h-1}+\ldots+jd^h. \end{aligned} \tag{3}$$

Hence

$$\sum_{r=1}^{t-1}\binom{k+t}{r}d^r l_{k+t-r} = l'_{k+t}-l_{k+t}-jd^{k+t}-\sum_{s=1}^{k}\binom{k+t}{k+t-s}d^{k+t-s}l_s. \tag{4}$$

Since $\{1, 2,\ldots, k\} \to k+t$, l'_{k+t} is a one-valued function of $l'_1, l'_2,\ldots, l'_k$, and so, by (3), a one-valued function of $d, l_1,\ldots, l_k$. Hence the right-hand side of (4) is a one-valued function of $d, l_1,\ldots, l_k$.

If we put $d = 1,\ldots, t-1$ in succession in (4) we have $t-1$ equations to determine $l_{k+1},\ldots, l_{k+t-1}$. For a given set $l_1,\ldots, l_k$ these equations have only one solution, provided that the determinant of the coefficients of $l_{k+t-1},\ldots, l_{k+1}$ on the left-hand side does not vanish. For this it is sufficient that D_{t-1} does not vanish, where

$$D_r = \begin{vmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & 2 & 2^2 & . & . & . & 2^{r-1} \\ 1 & 3 & 3^2 & . & . & . & 3^{r-1} \\ . & . & . & . & . & . & . \\ 1 & r & r^2 & . & . & . & r^{r-1} \end{vmatrix}.$$

But $$D_r = 2^{r-2}3^{r-3}\ldots(r-2)^2(r-1),$$

and so $D_{t-1} \neq 0$. Hence $l_{k+1},\ldots,l_{k+t-1}$ are one-valued functions of $l_1,\ldots,l_k$, that is,

$$\{1, 2,\ldots, k\} \to \{k+1, k+2,\ldots, k+t-1\}.$$

LEMMA 5. *If* $\{1, 2,\ldots, k\} \to k+1$, *then*

$$\{1, 2,\ldots, rk\} \to \{1, 2,\ldots, r(k+1)\}.$$

This is an immediate consequence of Lemmas 3 and 4.

The proof of Theorem 2 is now easy. Let us assume that

$$\{1, 2,\ldots, k\} \to k+1.$$

Putting $r = k+s-1$ in Lemma 5, we have

$$\{1, 2,\ldots, k(k+s-1)\} \to \{1, 2,\ldots, (k+1)(k+s-1)\} \qquad (s \geqslant 1).$$

Since $(k+1)(k+s-1) \geqslant k(k+s)$, the right-hand member of this relation for any s extends at least as far as the left-hand member of the relation for $s+1$. Hence

$$\{1, 2,\ldots, k^2\} \to \{1, 2,\ldots, j\}.$$

But, if $j = N(k^2)$, this contradicts Lemma 2. Hence, if $j = N(k^2)$,

$$\{1, 2,\ldots, k\} \nrightarrow k+1,$$

and so, by Lemma 1, $M(k) \leqslant N(k^2)$.

4. Further results.

THEOREM 4. *For every* k *there exists a* k' *such that* $k \leqslant k' < 2k$ *and* $M(k') = N(k') = N(k)$.

From this we have

THEOREM 5. $M(k) = N(k)$ *for infinitely many* k.

Theorem 4 is clearly an immediate consequence of

LEMMA 6. *If* $j \geqslant N(k)$, *then* $\{1,\ldots, k\} \nrightarrow \{k+1, k+2,\ldots, 2k\}$.

If $\{1, 2,\ldots, k\} \to \{k+1, k+2,\ldots, 2k\}$, then, by Lemma 3,

$$\{2, 4,\ldots, 2k\} \to \{2k+2, 2k+4,\ldots, 4k\},$$

and so, by Lemma 4,

$$\{1,\ldots, k\} \to \{1, 2,\ldots, 4k\}.$$

The process may be repeated and eventually we have

$$\{1,\ldots, k\} \to \{1, 2,\ldots, j\}.$$

But, by Lemma 2, this is impossible if $j \geqslant N(k)$.

5. Consequences in the 'easier' Waring's Problem.

Let $v(k)$ denote the least value of s such that every n is expressible as the sum of at most s numbers of the set

$$\pm m^k \qquad (m = 1, 2, \ldots).$$

Then I have proved (Wright, **7**) that

$$v(k) \leqslant 2M(k-2)+\Delta(k),$$

where $\Delta(k)$ is the least value of s such that every arithmetic progression contains a number which is the sum of at most s members of the set (5). I hope shortly to publish results for $\Delta(k)$ analogous to those of Hardy and Littlewood (**3**) for their function $\Gamma(k)$ in Waring's Problem. Hence I content myself with stating that

$$\Delta(k) \leqslant \tfrac{1}{2}(3k-1) \quad (k \text{ odd}), \qquad \Delta(k) \leqslant 2k \quad (k \text{ even}).$$

We at once deduce from Theorem 3

THEOREM 6.

$$v(k) \leqslant (k-2)^4+\tfrac{1}{3}(3k+5) \qquad (k \text{ odd}),$$
$$v(k) \leqslant (k-2)^4+2k \qquad (k \text{ even}).$$

From Theorem 5 we can deduce

THEOREM 7. $v(k) \leqslant k^2-\frac{5}{2}k+\frac{9}{2}$ *for an infinity of* k.

Theorems 6 and 7 are not new, since better results can be obtained from Vinogradoff's recent results (**6**) in the ordinary Waring's Problem. But Vinogradoff's proof is of a much less elementary character.

[*Note added* 2 *August* 1935. Dr. Chowla has drawn my attention to a proof of the result $N(k) \leqslant \frac{1}{2}(k^2+k+2)$ which he has published (**9**). As the proof I refer to above is shorter than his, I give it here, although, of course, Theorem 1 is a better result.

Let $j = \frac{1}{2}k(k+1)+1$, $n \geqslant (j!)j^k$, $1 \leqslant a_r \leqslant n$ $(1 \leqslant r \leqslant j)$. Then there are at least $n^j(j!)^{-1}$ sets $a_1, \ldots, a_j$ such that no set is a permutation of any other. If

$$l_h = a_1^h+\ldots+a_j^h,$$

then $j \leqslant l_h \leqslant jn^h$, and so there are at most

$$\prod_{h=1}^{k} (jn^h-j+1) < j^k n^{\frac{1}{2}k(k+1)} \leqslant n^j(j!)^{-1}$$

sets $l_1, \ldots, l_k$. Hence at least one set $l_1, \ldots, l_k$ corresponds to two sets $a_1, \ldots, a_j$ not permutations of one another; that is $N(k) \leqslant j$.

We observe that this proof (like that of Theorem 1) gives a definite upper bound for the a.]

REFERENCES

1. S. Chowla, *Proc. Indian Acad. Sci.* 1 (1935), 528–30.
2. Dickson, *History of the Theory of Numbers*, 2 (1924), ch. 24.
3. Hardy and Littlewood, *Proc. London Math. Soc.* (2) 28 (1928), 518–42.
4. Pillai, *Math. Student* (Madras), Sept. 1934.
5. Tarry, *L'intermédiaire des mathématiciens*, 19 (1912), 200, 219–21; 20 (1913), 68–70.
6. Vinogradoff, *C.R. Acad. Sci.* U.R.S.S. 4 (1934), 249–53.
7. Wright, *J. of London Math. Soc.* 9 (1934), 267–72.
8. ——, ibid. 10 (1935), 94–9.
9. S. Chowla, *Proc. Indian Acad. Sci.*, 1 (1935), 701–66.

A CERTAIN OPEN MANIFOLD WHOSE GROUP IS UNITY

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1. In a recent paper,† which will be referred to as T.M., I attempted unsuccessfully‡ to prove that every finite, unbounded manifold in which every circuit bounds a 2-cell is a 3-sphere. On the basis of T.M., Theorem 1, which is false, I had prepared an article containing the theorem on infinite manifolds announced in T.M., § 1. An obvious corollary to this theorem is that any infinite, unbounded manifold,‖ in which every finite circuit bounds a 2-cell and every finite 2-cycle bounds a finite region, is what we shall call a *formal 3-cell*. In § 3 below, an example is given which disproves this.

By a formal 3-cell we shall mean an infinite, unbounded manifold, a sub-division†† of which, say C, contains an infinite sequence of elements E_1, E_2,..., such that E_{n+1} contains every solid of C which meets E_n. Under these conditions it is obvious that any solid, and therefore any finite region, in C is contained in E_n for some value of n. Moreover, it is not difficult to show that a subdivision of C has a rectilinear model covering Euclidean 3-space; also that the symbol for such a rectilinear complex is a formal 3-cell. Thus an infinite manifold is a formal 3-cell, if and only if its rectilinear model in Hilbert space is in (1, 1) semilinear correspondence with Euclidean 3-space.

To avoid verbal complications we shall not always distinguish in our notation or terminology between a manifold and one of its subdivisions. Thus a manifold will mean an abstraction determined by the totality of symbols which are combinatorially equivalent‡‡ to a given symbolic manifold. Any symbolic manifold may be called a

† *Quart. J. of Math.* (Oxford), 5 (1934), 308–20. ‡ Ibid. 6 (1935).

‖ As in T.M., the words 'three-dimensional' will often be omitted.

†† In referring to a subdivision of an infinite manifold it is implied that any finite sub-complex has a finite subdivision. The theorems about combinatorial subdivisions which are relevant to this paper are to be found in articles by M. H. A. Newman (*J. of London Math. Soc.* 2 (1927), 56–64), and J. H. C. Whitehead (*Proc. Cambridge Phil. Soc.* 31 (1935), 69–75).

‡‡ These definitions, and many of the subsequent arguments, are based upon an article by J. W. Alexander, *Annals of Math.* 31 (1930), 292–320.

covering of the corresponding manifold, and in dealing with a sub-complex K, of a manifold M, we restrict ourselves to coverings which contain K as a sub-complex. More precisely, the transformation $(A, a)^{-1}$ shall not be applied to a covering of M unless $a(A)B$ belongs to K, where B is any component, including 1, such that aB belongs to K,† and (U) stands for the boundary of a complex U.

2. By a non-singular deformation of an i-dimensional manifold N_i ($i = 1$, 2, or 3), in a three-dimensional manifold M, we shall mean the resultant of a finite sequence of transformations of the form‡

$$N_i \to N_i + (E_{i+1})$$

if $i = 1$ or 2, where E_{i+1} is an $(i+1)$-element which meets N_i in an i-element on (E_{i+1}); and of the form

$$N_i \to N_i + E_i$$

if $i = 3$, where E_i is a 3-element whose boundary meets (N_i) in a 2-element and which is either contained in N_i or has no internal component in common with N_i.

LEMMA 1. *If a circuit c, on a two-dimensional manifold S, in M, is transformable into c' by a non-singular deformation, there is a non-singular deformation of S which carries c into c'.*

Let

$$c \to c + (E_2)$$

be the first step in the deformation of c. Let (E_2) meet c in a segment l and let

$$m = (E_2) - l.$$

After a slight deformation, the intersection $S.E_2$, if it exists, will consist of non-singular circuits and segments, the latter having their end-points on m, and S will not touch E_2, except along the boundary segment l. At least one of these circuits, say c_1, can be joined to a vertex on m by a non-singular segment t, on E_2, which does not meet S except in the end-point on c_1. Let p be this end-point and let

$$N(K, L)$$

stand for the aggregate of components in any symbolic complex L which meet a sub-complex K. With a suitable covering, $N(t, M)$ is a 3-element and $N(p, S)$ a 2-element which divides $N(t, M)$ into two

† Unity stands for the 'empty' complex, and the convention is that $L1 = L$, L being any complex.

‡ Addition is to modulus 2 throughout this paper, though we shall sometimes use the minus sign.

3-elements. Let E_3 be the one which contains t. Then c_1 is transformed into a segment by the deformation

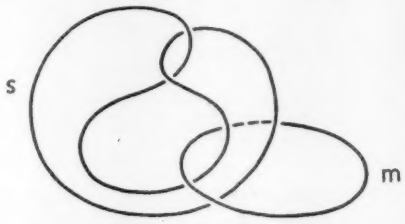
$$S \to S + (E_3).$$

In this way all the circuits in $S.E_2$ can be eliminated.

When the circuits have been eliminated there will be at least one segment in $S.E_2$, say u, which, together with a segment of m, bounds a 2-element C_2, on E_2, containing no other component of S. If E_3 is defined as before, with $N(C_2, M)$ and $N(u, S)$ taking the place of $N(t, M)$ and $N(p, S)$, the segment u is eliminated from $S.E_2$ by the deformation

$$S \to S + (E_3).$$

Reiterating this process, we obtain a non-singular deformation of



S into a surface S^1, which does not meet E_2 except in l, and this deformation leaves c unaltered. It is now obvious that the first step in the deformation $c \to c'$ can be realized by a non-singular deformation of S^1, and the lemma follows from induction on the number of steps in $c \to c'$.

If a circuit in M is contained in a 3-element in M it will be called an *elementary circuit*. A circuit which bounds a (singular) 2-cell but which is not an elementary circuit will be called a *self-linking* circuit. The simplest type of self-linking circuit is illustrated by the diagram, the manifold being the residual space of a circuit m in Euclidean space, and s being a self-linking circuit.

We shall need two lemmas about punctured spheres.† We first recall from T.M. pp. 319–20, that any two punctured spheres are equivalent if they have the same number of boundary 2-spheres.‡

† Cf. T. M. § 2. When we refer to a punctured sphere or to any other bounded manifold, it is to be assumed that the boundary is non-singular.

‡ The argument used in T.M. is valid in virtue of Alexander's theorem about the separation of a 3-sphere by a 2-sphere (*Proc. National Ac. of Sci.* 10 (1924), 6–8).

LEMMA 2. *Any complex K, in a punctured sphere U, is contained in an element which is contained in U, provided K does not separate any pair of 2-spheres in (U).*

The 2-spheres in (U) can be connected by a system of tubes which do not meet K, in such a way as to form a single 2-sphere which bounds a 3-element contained in U and containing K.

LEMMA 3. *Let U be a punctured sphere and E a 3-element whose boundary meets (U) in a band bounded by a pair of non-singular and non-intersecting circuits, and either*

(i) *let E have no other component in common with U,*

or (ii) *let E be contained in U.*

Then $U+E$ is a punctured sphere (case i) *or a pair of punctured spheres* (case ii).

Taking U to be actually imbedded in a 3-sphere the proof of the second case is obvious. In the first case one can, by the standard method of starring elements, imbed E in the 3-element bounded by the 2-sphere in (U) which meets (E), and the proof is again obvious.

From Lemma 1 it follows that any circuit which is contained in a punctured sphere in a given manifold is an elementary circuit.

Now let σ be a non-singular circuit in a manifold M, and let σ bound a 2-cell ϵ_2 which crosses itself along a non-singular segment $\alpha\beta$, α and β being vertices on σ. The non-singular image e_2, of ϵ_2, is in $(1, 1)$ correspondence with ϵ_2 except for two segments ab and $a'b'$, each of which corresponds to $\alpha\beta$. Let b and a' be inside e_2, and a and b' on s, the boundary of e_2 and the image of σ. Let x and x' be vertices on ab and $a'b'$ respectively, having the same image on $\alpha\beta$, and let t be any segment in e_2 joining x to x' and corresponding to a simple circuit τ, in M. If t does not meet ab or $a'b'$ except in x and x', we call it a *characteristic segment* and τ a *characteristic circuit* with respect to ϵ_2.

THEOREM 1. *If a given characteristic circuit is an elementary circuit, ϵ_2 is contained in a 3-element.*

Let a given characteristic circuit τ be contained in a 3-element. This 3-element can be deformed into one which contains τ and also the double segment $\alpha\beta$, and finally into a 3-element which contains ϵ_2, by the methods used in proving Lemma 1.

THEOREM 2. *If σ is an elementary circuit, a given characteristic circuit is either an elementary circuit or a one-sided circuit in a non-singular projective plane.*

In virtue of Lemma 2 it is sufficient to assume that σ is contained in a punctured sphere U, and we assume that no characteristic circuit is contained in a punctured sphere. If any characteristic circuit were contained in a punctured sphere, it would follow from Lemma 2 and Theorem 1 that any circuit on ϵ_2, treated as a complex in M, would be an elementary circuit. After a slight deformation of (U) we assume that no face of ϵ_2, nor edge of the double line $\alpha\beta$, lies on (U), and that each vertex of $(U).\epsilon_2$ is incident with precisely two edges of $(U).\epsilon_2$ if it does not lie on $\alpha\beta$, and with four edges if it lies on $\alpha\beta$. We also suppose that ϵ_2 does not touch (U), i.e. ϵ_2 crosses (U) at any common vertex.

Under these conditions the image of $(U).\epsilon_2$ in e_2 is a set of non-singular and non-intersecting circuits, which we shall call I. The vertices in which $\alpha\beta$ meets (U) will be called *double vertices*, and in the neighbourhood of a double vertex ξ, $(U).\epsilon_2$ corresponds to two segments in I meeting ab and $a'b'$ in x and x', the two images of ξ.

Our first step is to eliminate from $(U).\epsilon_2$ all the *isolated circuits*, that is to say, those which do not contain a double vertex. Such a circuit is the image of a circuit in I which does not meet ab or $a'b'$. If there is such a circuit, there is at least one which bounds a 2-element containing inside it no component of I. Let there be such a circuit, and let E_2 be the image on ϵ_2 of the 2-element in question. Let V stand for that one of the two regions U and $M-U$ which contains E_2. With a suitable covering, $N(E_2, V)$ is a 3-element which, with U, satisfies the conditions of Lemma 3. Therefore

$$U+N(E_2, V)$$

is either a punctured sphere containing U or a pair of punctured spheres, one of which contains σ. In either case U can be replaced by another punctured sphere containing σ such that the number of isolated circuits in $(U).\epsilon_2$ is reduced, and the number of double vertices is not increased. Without changing our notation we assume all the isolated circuits in $(U).\epsilon_2$ to have been eliminated in this way.

We proceed to replace the new region U by a punctured sphere such that no circuit in I meets ab or $a'b'$ more than once. No circuit in I separates b from a', since its image on (U) does not link σ. In the absence of isolated circuits, it follows that each circuit in I will then

meet each of ab and $a'b'$ just once. If there is a circuit in I which meets either ab or $a'b'$ more than once, I say that there is at least one segment l, on some circuit in I, which, together with a segment of ab or $a'b'$, bounds a 2-element E_2^0, containing no other component of I or of ab or $a'b'$. Let C_1 be any circuit in I which meets ab or $a'b'$, say ab, more than once. The 2-element bounded by C_1 either contains both b and a' or neither. First assume that it contains neither. Then the segments of ab and $a'b'$ which are inside this 2-element have both ends on C_1, and at least one of these segments, together with an arc of C_1, bounds a 2-element C_2, containing no other component of ab or of $a'b'$. The existence of the required 2-element E_2^0, follows from a similar argument, applied to the segments of I lying in C_2.

If the 2-element bounded by C_1 contains both b and a', let x be the intersection of ab with C_1 which is nearest to b, and y the next nearest. Then the segment xy of ab, together with one of the two arcs into which C_1 is separated by x and y, bounds a 2-element containing neither b nor a'. The existence of the 2-element E_2^0 now follows from the argument used in the previous case, the 2-element bounded by the two segments xy playing the part of the original 2-element bounded by C_1.

Let E_2 and λ be the images of E_2^0 and l in M, and let V stand for whichever of U or $M-U$ contains E_2. Then the double vertices of $(U).\epsilon_2$ in which $\alpha\beta$ meets λ are eliminated by means of the deformation

$$U \to U+N(E_2, V),$$

and no new double vertices are introduced. Any isolated circuits which are created by this deformation can be eliminated as before without increasing the number of double vertices. It follows from induction on the latter that σ is contained in a punctured sphere U such that each circuit in I meets each of the segments ab and $a'b'$ just once.

Now let $c_1, c_2, \ldots, c_k$ be the circuits in I, and let c_i meet ab in p_i and $a'b'$ in q_i. We suppose c_{i+1} to be contained in the 2-element bounded by c_i, so that p_{i+1} lies between p_i and b and q_{i+1} between q_i and a'. The vertices p and q occur in pairs, the vertices in each pair having the same image on $\alpha\beta$. From the order in which they occur it follows that

$$q_i = p'_{k-i+1}$$

where p_j and p'_j have the same image.

Since we are assuming that a characteristic circuit for ϵ_2 is not an elementary circuit, the set I is not empty and $k > 0$. Moreover k is even, say $k = 2r$, since α and β are both inside U. Let C_2 be one of the 2-elements into which the band bounded by c_r and c_{r+1} is separated by the segments $p_r p_{r+1}$ of ab and $q_r q_{r+1}$ of $a'b'$. Since

$$q_r = p'_{r+1}, \qquad q_{r+1} = p'_r,$$

the image of C_2 in M is a non-singular Möbius band μ_2, with its boundary (to modulus 2) on (U) and no other component on (U). A segment in C_2 joining a vertex on $p_r p_{r+1}$ to the corresponding vertex on $p'_r p'_{r+1}$ determines a characteristic circuit in M which is a one-sided circuit on μ_2. A 2-element on (U) bounded by (μ_2) can be added to μ_2 to provide a non-singular projective plane on which a characteristic circuit is a one-sided circuit.

From an argument similar to the one which led to the 2-cell E^0_2, bounded by l and a segment of ab or $a'b'$, it follows that any characteristic circuit can be transformed into any other by a non-singular deformation. Therefore the theorem follows from Lemma 1.

COROLLARY. *In a manifold with no torsion, σ is an elementary circuit, if and only if a given characteristic circuit is an elementary circuit.*

We conclude this section with a theorem which is not needed for the subsequent sections, but which may be of some general interest. Let $\sigma^1, \ldots, \sigma^n$ be any set of non-singular circuits in M, each of which bounds a 2-cell, and let each circuit σ^i contain at least one segment m^i, which does not belong to any of the others. Then the theorem is:

THEOREM 3. *The circuits σ bound a set of 2-cells whose intersections with themselves and each other consist of non-singular double segments joining vertices on $\sigma^1, \ldots, \sigma^n$.*

Let $\sigma^i = (\epsilon^i_2)$. Each of the 2-cells ϵ can obviously be deformed into a 2-cell which has no segment in common with any of the circuits σ, which does not touch any of these circuits, and which does not intersect σ^i except in m^i. After further slight deformations it may be assumed that $\epsilon^1_2, \ldots, \epsilon^n_2$ intersect themselves and each other at most in a set of double segments, at which two sheets belonging to ϵ^i_2 and ϵ^j_2 cross ($i = j$; or $i \neq j$), and triple points at which three sheets cross.† The branch points may be eliminated by Dehn's method, or by the similar method of cutting used in T.M. for the reduction of the model R.

† Cf. M. Dehn, *Math. Annalen*, 69 (1910), 147.

Each double line corresponds to two lines on e_2^i and e_2^j respectively ($i = j$; or $i \neq j$), e_2^k being the non-singular image of ϵ_2^k. It is either a segment (possibly singular) or a circuit. Let α, on one of the arcs m, be an end-point of a double segment which contains a triple point. Taking α as the first point of the double segment, let β be the first triple point, and let the sheet which cuts the double segment at β be on ϵ_2^k. Then the triple point β is eliminated by the deformation

$$\epsilon_2^k \to \epsilon_2^k + (E_3),$$

where E_3 is a 3-element defined, as in the proof of Lemma 1, in terms of $N(\alpha\beta, M)$ and the neighbourhood of β on the sheet which cuts $\alpha\beta$. No new triple points are created, and induction shows that all the triple points on the double segments can be eliminated.

The theorem now follows from induction on the number of double circuits and an argument similar to the one used in the proof of Lemma 1 to eliminate the circuits in $S.E_2$.

Any double segment γ corresponds to segments g^i and g^j on e_2^i and e_2^j. If g^i has one end on (e_2^i) and one end inside e_2^i, the same will be true of g^j with respect to e_2^j. In this case γ will be described as of the first type. Otherwise either g^i or g^j, say g^i, will have both its ends on (e_2^i), and g^j will have both ends inside e_2^j. If g^i and part of the segment on (e_2^i) corresponding to m^i bound a 2-element on e_2^i containing no other double segment, g^i can be eliminated by a deformation similar to the second deformation used in proving Lemma 1. In any case a deformation of the first kind, applied to the sheet g^i of ϵ_2^i, replaces γ by two double segments of the first type. For a vertex of g^j can be joined to a vertex on the image of m^j by a segment which does not meet any other double segment, and the image of this segment on ϵ_2^j can take the place of t in the proof of Lemma 1. Thus all the double segments may be replaced by double segments of the first type.

3. By a *ring* we shall mean an (orientable) anchor ring composed of two 3-elements E_1 and E_2, meeting in a pair of 2-elements common to (E_1) and (E_2), these 2-elements having no common vertex. The boundary of either 2-element will be called a *meridian* circuit, and a circuit on the boundary of the ring which intersects a meridian in a single vertex will be called a *longitudinal* circuit, or a *longitude*. We shall also describe as a longitude any circuit in the ring which is isotopic to a longitudinal circuit on the boundary.

LEMMA 4. *Either of two symbolic rings with a common boundary can be transformed internally into the other provided they have a meridian circuit in common.*

Let R_1 and R_2 be the two rings and let

$$R_1 = E_1 + E_1^*,$$

where E_1 and E_1^* meet in 2-elements bounded by meridian circuits m and m^*, m being also a meridian of R_2. After a suitable internal subdivision $R_2 \to R_2'$, it is clear that

$$R_2' = E_2 + E_2^*,$$

where E_2 and E_2^* are 3-elements meeting in 2-elements bounded by m and m^*. Then R_1 and R_2' can be transformed internally into the same ring by starring first E_i and E_i^* $(i = 1, 2)$ and then the common 2-elements. Since $R_2 \to R_2'$ internally, the lemma is established.

Let R_1 be a ring contained in a symbolic manifold M, and let R_2 be a symbolic ring whose boundary is identical with (R_1), the two rings having a common meridian. Then from Lemma 4 we have

LEMMA 5. *If R_2 has no internal component in common with M,*

$$M \to M - R_1 + R_2$$

by transformations which are internal to R_1.

A circuit σ in a ring M will be called a *self-linking circuit of the first type* if it bounds a 2-cell of the kind described in the last section, with a longitude as a characteristic circuit. If M is taken to be the region outside an unknotted tube m, in a 3-sphere, the diagram in § 2 represents a self-linking circuit of the first type. If $n > 1$, σ will be called a self-linking circuit of the nth type if it bounds a 2-cell of the kind described in § 2, with a self-linking circuit of the $(n-1)$th type as a characteristic circuit. From the corollary to Theorem 2 and induction on n it follows that a self-linking circuit of the nth type is self-linking in the sense of § 2.

Let s be a self-linking circuit of the first type inside a symbolic ring R. After a suitable subdivision, $N(s, R)$ will be a ring S, any longitude in which will be a self-linking circuit of the first type in R. Let m be a meridian circuit and l a longitude on (S), l and m having a single vertex in common. Let M and L respectively be meridian and longitudinal circuits on (R), meeting in a single vertex.

After a suitable subdivision† of R we may suppose (R) and (S) to

† i.e. a general subdivision (Newman, loc. cit.).

be congruent, L and M corresponding to l and m respectively. Without altering our notation we suppose this to be the case.

Let $$U = P(a_1, a_2, \ldots, a_N)$$

be the region $R-S$, and let

$$U^i = P(a_1^i, a_2^i, \ldots, a_N^i) \qquad (i = 1, 2, \ldots)$$

be an infinite sequence of copies of U, P being the same function in each case. Let

$$(R^i) = F(b_1^i, \ldots, b_k^i), \qquad (S^i) = F(c_1^i, \ldots, c_k^i),$$

the b's and c's being certain of the vertices a, and the congruence $(R) \equiv (S)$ being given by

$$b_\lambda \to c_\lambda.$$

Writing b_λ^0 for b_λ, let b_λ^{i-1} be substituted for c_λ^i in U^i $(\lambda = 1, \ldots, k;$ $i = 1, 2, \ldots)$, and let V^i be the region into which U^i is thus transformed. Then I say that the manifold

$$W = R + \sum_{i=1}^{\infty} V^i$$

satisfies the conditions

(i) *every circuit bounds a* 2*-cell*;

(ii) *every finite* 2*-cycle bounds a finite region*;

(iii) *every non-singular* 2*-sphere bounds a* 3*-element*;

and that W is not a formal 3-cell.

For let $$W^n = R + \sum_{i=1}^{n} V^i,$$

with $W^0 = R$. Then it follows from induction on n and Lemma 5 that W^n is a ring for every value of n, a meridian circuit on (W^n) $((W^n) = (R^n))$ corresponding to a meridian on (S^{n+1}) in the congruence

$$b_\lambda^n \to c_\lambda^{n+1}.$$

The conditions (ii) and (iii) are satisfied by any ring. Therefore they are satisfied by W, since any finite region in W is contained in W^n for some value of n.

Let l^n be the longitude on (S^n) which corresponds to l on (S). Then l^n bounds a 2-cell in W^{n+1}, since any longitude in S is deformable into s, and s bounds a 2-cell in R. Therefore any circuit in W^n bounds a 2-cell in W^{n+1}, and it follows that the manifold W satisfies the first, and therefore all the conditions (i), (ii), (iii).

Since $$R^n - S^n = U^n \to V^n = W^n - W^{n-1}$$

on replacing c_λ^n by b_λ^{n-1}, it follows from Lemma 5 that $R^n \to W^n$ by

a series of simple transformations which, except for the substitutions $(c_\lambda^n, b_\lambda^{n-1})$, are internal to S^n. It follows that R^n and W^n have a common subdivision which, except for the substitutions $(c_\lambda^n, b_\lambda^{n-1})$, leaves U^n and V^n unaltered.† Since l is a self-linking circuit of the first type in R, it follows that L^{n-1} is a self-linking circuit of the first type in W^n, L^i being the circuit on (R^i) corresponding to L on (R). Therefore any longitudinal circuit in W^{n-1} is a self-linking circuit of the first type in W^n. It follows from induction on k that any self-linking circuit of the kth type in W^{n-1} is a self-linking circuit of the $(k+1)$th type in W^n. From induction on p it follows that any self-linking circuit of the kth type in W^n is a self-linking circuit of the $(k+p)$th type in W^{n+p}. In particular, l is a self-linking circuit of the nth type in W^n.

If W were a formal 3-cell, some subdivision of l would be contained in an element which would itself be contained in a subdivision of W^n for some value of n. This would contradict the fact that l is a self-linking circuit in W^n. Therefore W is not a formal 3-cell.

4. Leaving aside the question whether or no all the spaces defined by the methods of § 3 are equivalent, we shall show that one of them has a semilinear map in a semilinear 3-sphere.‡

By an unknotted ring R, in a 3-sphere H, we shall mean one such that $H-R$ is a ring. With a suitable covering it is clear that the neighbourhood of an unknotted circuit is an unknotted ring.

Let R be an unknotted ring in a 3-sphere H, which we take to be a rectilinear simplicial complex in Euclidean N-space $(N > 3)$. Let s be an unknotted, self-linking circuit of the first type in R (cf. the diagram in § 2), and let W be the manifold defined by means of R, s and the construction given in § 3. We shall take W to be a rectilinear complex in Hilbert space, with the symbol given in § 3. Since s is unknotted, $H-S$ is a ring, and we assume the congruence $(R) \equiv (S)$ of § 3 to be such that meridian circuits of R and $H-R$ correspond to meridians of S and $H-S$, respectively.

Let T^n be an unknotted ring in H, and let W^n be mapped semilinearly on $H-T^n$ in such a way that the image in the congruence $(R) \equiv (W^n)$ of a meridian on $H-R$ is mapped on a meridian of T^n. Then the initial congruence $(R) \equiv (S)$ determines a semilinear mapping $(S) \to (T^n)$, such that a meridian of $H-S$ corresponds to

† J. H. C. Whitehead, *Proc. Cambridge Phil. Soc.* loc. cit.

‡ Cf. J. H. C. Whitehead, *Proc. National Ac. of Sci.*, 21 (1935), 364–6.

a meridian of T^n. From Lemma 5 it follows that $H-S$ can be mapped semilinearly on T^n in such a way that the transformation $H-S \to T^n$ coincides with $(S) \to (T^n)$ on (S). Let T^{n+1} be the image of $H-R$ in $H-S \to T^n$. Then T^n-T^{n+1} is the image of $R-S$, and $W^{n+1}-W^n$ is mapped semilinearly on T^n-T^{n+1} in such a way that the transformations

$$W^n \to H-T^n, \qquad W^{n+1}-W^n \to T^n-T^{n+1}$$

can be united to form a semilinear transformation

$$W^{n+1} \to H-T^{n+1}.$$

Moreover a meridian of $H-R$, and therefore the corresponding circuit of W^{n+1}, is mapped on a meridian of T^{n+1}.

If $T^0 = H-R$, an infinite sequence of rings T^0, T^1,... $(T^n \subset T^{n+1})$ is defined inductively, and W is mapped semilinearly on $H-X$, where X is the set of points common to T^0, T^1,....

THE SURFACE OF AN ELLIPSOID

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So far as I know, it has never hitherto been noticed that the surface of an ellipsoid, *qua* function of the semi-axes, is a solution of a partial differential equation of the first order of Lagrange's linear type in which the coefficients are rational functions of the semi-axes. I encountered this differential equation quite naturally when endeavouring to solve the problem of determining the ellipsoidal conductor of given surface and maximum or minimum capacity.

The result seems to me to be of sufficient interest to justify its being put on record, and I give two proofs of it in this paper. Although the analysis in the first proof is unsymmetrical, it is the method by which I constructed the equation and it seems to be a more natural method of approaching the equation than the symmetrical verification which constitutes my second proof.

I conclude the paper by constructing a general integral of the partial differential equation in question.

1. Let a, b, c be the semi-axes of the ellipsoid arranged in descending order of magnitude. Let k and k' be the positive numbers such that

$$k^2 = \frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}, \qquad k'^2 = \frac{c^2(a^2-b^2)}{b^2(a^2-c^2)},$$

and let γ be the positive acute angle such that

$$c = a\cos\gamma,$$

so that

$$b = +\frac{a\cos\gamma}{\sqrt{(1-k^2\sin^2\gamma)}}.$$

Also let*

$$\Delta(\theta, k) = +\sqrt{(1-k^2\sin^2\theta)},$$

$$E(\gamma, k) = \int_0^\gamma \Delta(\theta, k)\, d\theta, \qquad F(\gamma, k) = \int_0^\gamma \frac{d\theta}{\Delta(\theta, k)}.$$

Then Legendre's formula† for the surface of an ellipsoid is

$$S = 2\pi a^2\cos^2\gamma + \frac{2\pi a^2\cot\gamma}{\Delta(\gamma, k)}\{E(\gamma, k)\sin^2\gamma + F(\gamma, k)\cos^2\gamma\}.$$

* This is Legendre's original notation. It is now usual to write $E(u, k)$ for what was formerly called $E(\mathrm{am}\, u, k)$.

† A. M. Legendre, *Exercices de calcul intégral*, 1 (1811), 182–94; *Traité des fonctions elliptiques*, 1 (1825), 350–60.

At this stage I remark that the well-known formulae

$$k\frac{\partial E(\gamma,k)}{\partial k} = E(\gamma,k)-F(\gamma,k),$$

$$k\frac{\partial F(\gamma,k)}{\partial k} = \frac{E(\gamma,k)}{k'^2}-F(\gamma,k)-\frac{k^2\sin\gamma\cos\gamma}{k'^2\Delta(\gamma,k)},$$

$$\frac{\partial E(\gamma,k)}{\partial\gamma} = \Delta(\gamma,k), \qquad \frac{\partial F(\gamma,k)}{\partial\gamma} = \frac{1}{\Delta(\gamma,k)}$$

make it possible to express not only S but also $\partial S/\partial k$ and $\partial S/\partial\gamma$ as linear combinations of the incomplete elliptic integrals $E(\gamma,k)$ and $F(\gamma,k)$. The possibility of eliminating the two elliptic integrals from these three formulae is the source of the partial differential equation which will ultimately emerge.

After this digression we proceed with the investigation. Write for brevity

$$\frac{E(\gamma,k)\sin^2\gamma+F(\gamma,k)\cos^2\gamma}{\sin\gamma} = \Theta,$$

so that

$$S = 2\pi c^2+2\pi ab\Theta.$$

We then have immediately

$$\frac{\partial\Theta\sin\gamma}{\partial\gamma} = 2\sin\gamma\cos\gamma\{E(\gamma,k)-F(\gamma,k)\}+\frac{1-k^2\sin^4\gamma}{\Delta(\gamma,k)},$$

$$k\frac{\partial\Theta\sin\gamma}{\partial k} = \sin^2\gamma\{E(\gamma,k)-F(\gamma,k)\}+$$
$$+\cos^2\gamma\left\{\frac{E(\gamma,k)}{k'^2}-F(\gamma,k)-\frac{k^2\sin\gamma\cos\gamma}{k'^2\Delta(\gamma,k)}\right\},$$

where γ and k are regarded as the independent variables.

The result of eliminating the incomplete elliptic integrals from the three equations containing Θ and its differential coefficients is easily seen to be

$$k^2\Theta\sin\gamma\cos^2\gamma+\frac{k'^2+k^2\cos^4\gamma}{2\sin\gamma\cos\gamma}\frac{\partial\Theta\sin\gamma}{\partial\gamma}-kk'^2\sin\gamma\frac{\partial\Theta}{\partial k}$$
$$=\frac{k^2\sin\gamma\cos^3\gamma}{\Delta(\gamma,k)}+\frac{(1-k^2\sin^4\gamma)(k'^2+k^2\cos^4\gamma)}{2\sin\gamma\cos\gamma\Delta(\gamma,k)},$$

that is to say,

$$\left(k^2\cos^2\gamma+\frac{k'^2+k^2\cos^4\gamma}{2\sin^2\gamma}\right)\Theta+\frac{k'^2+k^2\cos^4\gamma}{2\sin\gamma\cos\gamma}\frac{\partial\Theta}{\partial\gamma}-kk'^2\frac{\partial\Theta}{\partial k}$$
$$=\frac{k^2\cos^3\gamma}{\Delta(\gamma,k)}+\frac{(1-k^2\sin^4\gamma)(k'^2+k^2\cos^4\gamma)}{2\sin^2\gamma\cos\gamma\Delta(\gamma,k)}.$$

We now change the independent variables from γ, k to b, c. Since

$$\frac{\partial}{\partial\gamma}=\frac{\partial b}{\partial\gamma}\frac{\partial}{\partial b}+\frac{\partial c}{\partial\gamma}\frac{\partial}{\partial c}, \qquad \frac{\partial}{\partial k}=\frac{\partial b}{\partial k}\frac{\partial}{\partial b},$$

we have

$$\frac{\partial}{\partial\gamma}=-\frac{ak'^2\sin\gamma}{\Delta^3(\gamma,k)}\frac{\partial}{\partial b}-a\sin\gamma\frac{\partial}{\partial c}, \qquad \frac{\partial}{\partial k}=\frac{ak\sin^2\gamma\cos\gamma}{\Delta^3(\gamma,k)}\frac{\partial}{\partial b},$$

so that

$$\frac{k'^2+k^2\cos^4\gamma}{2\sin\gamma\cos\gamma}\frac{\partial\Theta}{\partial\gamma}-kk'^2\frac{\partial\Theta}{\partial k}$$
$$=-\frac{a(k'^2+k^2\cos^4\gamma)}{2\cos\gamma}\frac{\partial\Theta}{\partial c}-\frac{ak'^2(1-k^2\sin^4\gamma)}{2\cos\gamma\,\Delta^3(\gamma,k)}\frac{\partial\Theta}{\partial b},$$

and therefore Θ satisfies the partial differential equation

$$\frac{1-k^2\sin^4\gamma}{2\sin^2\gamma}\Theta-\frac{ak'^2(1-k^2\sin^4\gamma)}{2\cos\gamma\,\Delta^3(\gamma,k)}\frac{\partial\Theta}{\partial b}-\frac{a(k'^2+k^2\cos^4\gamma)}{2\cos\gamma}\frac{\partial\Theta}{dc}$$
$$=\frac{k^2\cos^3\gamma}{\Delta(\gamma,k)}+\frac{(1-k^2\sin^4\gamma)(k'^2+k^2\cos^4\gamma)}{2\sin^2\gamma\cos\gamma\,\Delta(\gamma,k)}.$$

When the coefficients in this equation are expressed in terms of b and c, it becomes

$$c^2(a^2+b^2-c^2)\Theta-b(a^2-b^2)(a^2+b^2-c^2)\frac{\partial\Theta}{\partial b}-c(a^2-c^2)(a^2-b^2+c^2)\frac{\partial\Theta}{\partial c}$$
$$=\frac{c^2}{a^3b}\{2b^4(a^2-c^2)+a^2(a^2+b^2-c^2)(a^2-b^2+c^2)\}.$$

Replacing Θ by its value in terms of S, we get

$$\left[c^2(a^2+b^2-c^2)-b(a^2-b^2)(a^2+b^2-c^2)\frac{\partial}{\partial b}-c(a^2-c^2)(a^2-b^2+c^2)\frac{\partial}{\partial c}\right]\frac{S}{2\pi ab}$$
$$=2abc^2,$$

so that

$$(a^2-b^2+c^2)(a^2+b^2-c^2)S-b(a^2-b^2)(a^2+b^2-c^2)\frac{\partial S}{\partial b}-$$
$$-c(a^2-c^2)(a^2-b^2+c^2)\frac{\partial S}{\partial c}=4\pi a^2b^2c^2.$$

Now S is a homogeneous function of a, b, c, of degree two, and so, by Euler's theorem,

$$2S=a\frac{\partial S}{\partial a}+b\frac{\partial S}{\partial b}+c\frac{\partial S}{\partial c}.$$

When we substitute the expression on the right for $2S$, we obtain the differential equation in a symmetrical form, namely,

$$a(a^2-b^2+c^2)(a^2+b^2-c^2)\frac{\partial S}{\partial a}+b(b^2-c^2+a^2)(b^2+c^2-a^2)\frac{\partial S}{\partial b}+$$
$$+c(c^2-a^2+b^2)(c^2+a^2-b^2)\frac{\partial S}{\partial c}=8\pi a^2b^2c^2.$$

For brevity we shall write

$$a(a^2-b^2+c^2)(a^2+b^2-c^2)\equiv f(a;b,c),$$

so that the partial differential equation assumes the form

$$f(a;b,c)\frac{\partial S}{\partial a}+f(b;c,a)\frac{\partial S}{\partial b}+f(c;a,b)\frac{\partial S}{\partial c}=8\pi a^2b^2c^2,$$

i.e.
$$\sum_{a,b,c} f(a;b,c)\frac{\partial S}{\partial a}=8\pi a^2b^2c^2.$$

2. Now that the differential equation satisfied by the surface of an ellipsoid has actually been constructed in the direct but unsymmetrical manner which has just been given, it is possible to fabricate a verification of the result in which the working is symmetrical as regards the semi-axes.

An expression for the surface of an ellipsoid is

$$S=\pi a^2b^2c^2\int_0^\infty\left(\sum_{a,b,c}\frac{1}{a^2+\lambda}\right)\frac{d\lambda}{\sqrt{\{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}}.$$

This manifestly symmetrical formula is due to Edwards*; it is easily deducible from the formula involving Weierstrassian elliptic functions given by Tannery and Molk.†

It is merely a matter of somewhat tedious algebra to verify that

$$\left(\sum_{a,b,c}f(a;b,c)\frac{\partial}{\partial a}\right)\left[\left(\sum_{a,b,c}\frac{1}{a^2+\lambda}\right)\frac{a^2b^2c^2}{\sqrt{\{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}}\right]$$

is equal to

$$\frac{\partial}{\partial\lambda}\left[\frac{2a^2b^2c^2\sqrt{\lambda}}{\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}}\left(c_1\lambda+c_2+\frac{A}{a^2+\lambda}+\frac{B}{b^2+\lambda}+\frac{C}{c^2+\lambda}\right)\right],$$

where the constants A, B, C, c_1, c_2 are given by the formulae

$$c_1=4,\qquad c_2=2\sum_{a,b,c}a^2$$
$$A=-(a^2-b^2+c^2)(a^2+b^2-c^2),$$

* J. Edwards, *The Integral Calculus*, 1 (1921), 842–3.

† J. Tannery et J. Molk, *Théorie des fonctions elliptiques*, 4 (1902), 170–2.

with corresponding values for B and C. Hence

$$\sum_{a,b,c} f(a;b,c)\frac{\partial S}{\partial a}$$

$$= 2\pi a^2b^2c^2 \int_0^\infty \frac{\partial}{\partial\lambda}\left[\frac{\sqrt{\lambda}}{\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}}\left\{4\lambda+2\sum_{a,b,c}\left(a^2-\frac{f(a;b,c)}{a(a^2+\lambda)}\right)\right\}\right]d\lambda$$

$$= 8\pi a^2b^2c^2,$$

and this completes the verification.

3. Finally we construct a general integral of the partial differential equation

$$\sum_{a,b,c} a(a^2-b^2+c^2)(a^2+b^2-c^2)\frac{\partial z}{\partial a} = 8\pi a^2b^2c^2,$$

of which a particular solution is now known to be

$$z = S.$$

Put $z-S = w$ so that

$$\sum_{a,b,c} f(a;b,c)\frac{\partial w}{\partial a} = 0.$$

Lagrange's rule for obtaining a general integral of an equation of this linear type requires the construction of two independent integrals of the subsidiary system of ordinary differential equations

$$\frac{da}{f(a;b,c)} = \frac{db}{f(b;c,a)} = \frac{dc}{f(c;a,b)}$$

in the forms $\qquad u_1 = \alpha_1, \qquad u_2 = \alpha_2,$

where u_1, u_2 are definite functions (i.e. functions involving no arbitrary elements) of some or all of a, b, c, while α_1, α_2 are the constants of integration. A general integral of the partial differential equation is then

$$w = \Phi(u_1, u_2),$$

where Φ denotes an arbitrary function of the functions u_1, u_2.

To obtain the two integrals of the subsidiary system we naturally change the variables from a, b, c to a, γ, k (where γ and k have the meanings previously assigned to them); it is found without difficulty that the subsidiary system is equivalent to the pair of simultaneous ordinary differential equations

$$\frac{2da}{a} = \frac{\cot\gamma\, d\gamma}{1-k^2\sin^2\gamma} + \tan\gamma\, d\gamma,$$

$$-2kk'^2\sin\gamma\cos\gamma\, d\gamma = (k'^2+k^2\cos^4\gamma)\, dk.$$

The second of these equations does not involve a; it is an equation of Riccati's type to determine $\cos^2\gamma \equiv \xi$ in terms of k. We accordingly make the standard substitution

$$\xi \equiv \cos^2\gamma = -\frac{k'^2}{k}\frac{d\log\zeta}{dk},$$

and we find that ζ satisfies the equation

$$kk'^2\frac{d^2\zeta}{dk^2} - (1+k^2)\frac{d\zeta}{dk} + k\zeta = 0.$$

If K and E denote the complete elliptic integrals with modulus k, while K' and E' denote the corresponding integrals with the complementary modulus k', two independent integrals of the equation for ζ are

$$E', \qquad K-E,$$

and so the general solution of the Riccati equation is

$$\cos^2\gamma = -\frac{\alpha_1 E-(K'-E')}{\alpha_1(K-E)-E'},$$

where α_1 is the constant of integration. This solution may also be written in the form

$$\frac{K'-E'\sin^2\gamma}{K\cos^2\gamma+E\sin^2\gamma} = \alpha_1.$$

Now that this integral has been obtained, the simplest way of constructing the other integral which is required is to substitute for γ in terms of k and α_1 on the right in the equation

$$\frac{4\,da}{a} - 2\tan\gamma\,d\gamma = \frac{2\cot\gamma\,d\gamma}{1-k^2\sin^2\gamma},$$

so that we have an equation involving two variables only (namely, $a^2\cos\gamma$ and k), and, after this equation has been integrated, to substitute back for α_1 in terms of γ and k in order to obtain an equation involving one constant of integration only.

When the first of these substitutions has been effected, the variables are separated and we merely have to evaluate

$$\int \frac{2\cot\gamma\,d\gamma}{1-k^2\sin^2\gamma} \equiv -\int \frac{k'^2+k^2\xi^2}{kk'^2(1-\xi)(k'^2+k^2\xi)}\,dk,$$

where ξ is supposed to be expressed in terms of k and α_1.

Now

$$-\frac{k'^2+k^2\xi^2}{kk'^2(1-\xi)(k'^2+k^2\xi)} \equiv \frac{1}{kk'^2} - \frac{1}{kk'^2(1-\xi)} - \frac{1}{k(k'^2+k^2\xi)},$$

and, if we write

$$\alpha_1 K - K' \equiv \theta,$$

$$\alpha_1(k'^2K - E) + (k^2K' - E) \equiv \phi$$

for brevity, it is easy to verify that

$$\frac{1}{1-\xi} = \frac{\alpha_1(K-E)-E'}{\alpha_1 K - K'}$$

$$= k^2 - kk'^2 \frac{d\log\theta}{dk},$$

$$\frac{1}{k'^2+k^2\xi} = \frac{\alpha_2(K-E)-E'}{\alpha_2(k'^2K-E)+k^2K'-E'}$$

$$= 1 - k\frac{d\log\phi}{dk}.$$

Hence we have

$$\frac{4da}{a} - 2\tan\gamma\, d\gamma = \frac{d\theta}{\theta} + \frac{d\phi}{\phi},$$

and so the required integral is obtained in the form

$$\tfrac{1}{4}\pi^2 a^4 \cos^2\gamma = \alpha_2^2 \theta\phi,$$

where α_2 is the constant of integration; the object of inserting the factor $\frac{1}{4}\pi^2$ will be seen in a moment.

When we write out the product $\theta\phi$ in full in terms of k and α_1, and then substitute for α_1 in terms of γ and k, we easily find that

$$\theta\phi = \frac{\sin^2\gamma(1-k^2\sin^2\gamma)(EK'+E'K-KK')^2}{(K\cos^2\gamma+E\sin^2\gamma)^2}$$

$$= \frac{\tfrac{1}{4}\pi^2\sin^2\gamma(1-k^2\sin^2\gamma)}{(K\cos^2\gamma+E\sin^2\gamma)^2}$$

$$= \frac{\tfrac{1}{4}\pi^2 a^2c^2(a^2-c^2)}{b^2\{c^2K+(a^2-c^2)E\}^2},$$

the second step following from Legendre's relation. The second integral of the subsidiary system is therefore

$$\frac{b\{c^2K+(a^2-c^2)E\}}{\sqrt{(a^2-c^2)}} = \alpha_2.$$

Lagrange's rule now asserts that a general integral of the partial differential equation

$$\sum_{a,b,c} a(a^2-b^2+c^2)(a^2+b^2-c^2)\frac{\partial z}{\partial a} = 8\pi a^2b^2c^2$$

is

$$z - S = \Phi(u_1, u_2),$$

where Φ denotes an arbitrary function of its two variables, these variables being defined by the formulae

$$u_1 \equiv \frac{a^2K' - (a^2 - c^2)E'}{c^2K + (a^2 - c^2)E},$$

$$u_2 \equiv \frac{b\{c^2K + (a^2 - c^2)E\}}{\sqrt{(a^2 - c^2)}},$$

and S denotes the surface of the ellipsoid with semi-axes a, b, c; further, K and E are the complete elliptic integrals with modulus

$$k = \sqrt{\frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}},$$

and K' and E' are the complete elliptic integrals with the complementary modulus

$$k' = \sqrt{\frac{c^2(a^2 - b^2)}{b^2(a^2 - c^2)}}.$$

HYPERGEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS (I)

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1. Introduction

Partial differential equations of the type

$$\prod f_r(\delta_r)Z = (\prod x_r^{a_r}) \prod g_r(\delta_r)Z, \tag{1}$$

where $\delta_r \equiv x_r \partial/\partial x_r$, and f_r, g_r are polynomials with constant coefficients, are the natural extension, to a field of n independent variables $x_1,\ldots,x_n$, of the ordinary differential equations

$$f(\delta)y = x^a g(\delta)y. \tag{2}$$

These latter equations (2) are often known as 'hypergeometric differential equations' (in the wider sense),* and it seems not improper to extend the term to the partial equations (1).

For present purposes I impose certain restrictions on the generality of (1). In the first place I exclude any case $a_r = 0$, leaving it to be deduced as a limit $a_r \to 0$. We can then, by change of independent variable, reduce (1) to the simpler form

$$\prod f_r(\delta_r)Z = (x_1 \ldots x_n) \prod g_r(\delta_r)Z \tag{3}$$

in which every a_r is now unity. I exclude too the case in which any $f_r(\delta_r)$ has equal roots or roots differing by an integer. This corresponds to the case of a 'logarithmic' solution of an ordinary hypergeometric equation.†

Now let p_r, q_r denote the orders of the operators $f_r(\delta_r)$, $g_r(\delta_r)$, i.e. the degrees of the polynomials $f_r(t)$, $g_r(t)$; then, if *every* $p_r \geqslant q_r$ (or equivalently every $q_r \geqslant p_r$), I shall say that the equation (1) or (3) is of *hyperbolic* type: if, alternatively, some $p_r > q_r$, while another $q_s > p_s$, so that the operators of predominating order in the several independent variables are not all on the same side of the equation, then I shall say that the equation is of *parabolic* type. These are reasonable (if unorthodox) entensions from the fundamental hyperbolic equation

$$\frac{\partial^2 Z}{\partial x \partial y} = Z, \quad \text{i.e. } \delta\delta' Z = xyZ, \tag{4}$$

* I have discussed them at some length under this title in my *Differential Calculus* (Oxford, 1935), 421–34. † Chaundy, loc. cit., 427–34.

and the fundamental parabolic equation

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\partial Z}{\partial y}, \quad \text{i.e. } \delta(\delta-1)Z = x^2 y^{-1}\delta' Z, \tag{5}$$

where $\delta,\ \delta' \equiv x\partial/\partial x,\ y\partial/\partial y$. I exclude parabolic equations from this investigation and consider only equations of 'hyperbolic' type.

For simplicity I begin with the equation in two independent variables

$$f(\delta)F(\delta')Z = xy\,g(\delta)G(\delta')Z. \tag{6}$$

If p, q, P, Q are the degrees of f, g, F, G, we have in the hyperbolic case, which we are now considering, $p \geqslant q,\ P \geqslant Q$. Moreover, we may sufficiently suppose that the leading coefficients in these four polynomials are all unity.

I follow in essentials the procedure introduced by Riemann for the hyperbolic equations of the second order.* Associating the given equation $\Delta Z = 0$ (say) with its adjoint $\Delta' Z = 0$, we first seek the *fundamental solution* $U(x, y; X, Y)$ such that

(i) for x, y constant, U is a solution of $\Delta Z = 0$ in variables X, Y;

(ii) for X, Y constant, U is a solution of $\Delta' Z = 0$ in variables x, y;

(iii) on each of the 'lines' $x = X,\ y = Y$, U reduces to a constant independent of x, y, X, Y, which we may take to be unity.

By the defining property of the adjoint operator we can write, for some α, β,

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} = U\Delta Z - Z\Delta' U.$$

The right-hand side vanishes, if Z, U are respectively solutions of $\Delta Z = 0,\ \Delta' U = 0$. Thus, by Stokes's theorem

$$\int (\alpha\, dy - \beta\, dx) = 0,$$

taken round any simple closed curve. We take the *characteristics* $x = X,\ y = Y$ as part of this curve, completing it either (i) by a second pair of characteristics drawn through an arbitrary point, say the characteristics $x = 0,\ y = 0$ through the origin, or (ii) by the arc of a given curve C cut off between $x = X,\ y = Y$. In (i) we

* See, for instance, Darboux, *La Théorie générale des surfaces*, 2 (Paris, 1915), 71–111, chap. IV, or Goursat, *Cours d'analyse mathématique*, 3 (Paris, 1915), 147–58. For a discussion of Cauchy's problem see also Hadamard, *Lectures on Cauchy's Problem* (Yale, 1923), Book I.

get the *general* solution of ΔZ (in variables X, Y) in the form

$$Z(X,Y) = \int_0^X Z_x(x,0)U(x,0;X,Y)\,dx + \int_0^Y Z_y(0,y)U(0,y;X,Y)\,dy + \\ + Z(0,0)U(0,0;X,Y).$$

We may regard this as giving the 'general' solution in terms of two arbitrary functions

$$\phi(x) \equiv Z_x(x,0), \qquad \psi(y) \equiv Z_y(0,y)$$

and one arbitrary constant $C \equiv Z(0,0)$, or, alternatively, in terms of the values of $Z(x,y)$ and certain of its derivatives along the selected characteristics $x = 0$, $y = 0$. In the language of 'Cauchy's problem' we say that the 'data' are borne by these characteristics. In (ii) the data of the problem are borne, more generally, by the prescribed curve C.

2. The fundamental solution

Now we can form particular solutions of (6) in power-series in the way that we form the standard series-solutions of (2).* The series will proceed by powers of xy, and, for the ascending series, the first term must be annihilated by the 'indicial' operator $f(\delta)F(\delta')$. Thus, if a, A are typical zeros of f, F respectively, typical leading terms will be $x^a y^{A+k}$, $x^{a+h}y^A$, where h, k are arbitrary. The series led by $x^a y^{A+k}$ is found to be

$$\sum_{n=0}^{\infty} \frac{\prod_{t=0}^{n-1} g(a+t)G(A+k+t)}{\prod_{t=1}^{n} f(a+t)F(A+k+t)} x^{a+n}y^{A+k+n}. \tag{7}$$

Out of these elementary solutions can be constructed, as we shall see, the more elaborate solution

$$U(x,y;X,Y) \equiv \\ \sum_{n=0}^{\infty} \sum_{a} \sum_{A} \left(\frac{x}{X}\right)^a \left(\frac{y}{Y}\right)^A \sum_{r=0}^{n} \sum_{R=0}^{n} \frac{\prod_{t=0}^{n-1} g(a-r+t)G(A-R+t)}{\prod_{t=0}^{n}{}' f(a-r+t)F(A-R+t)} x^{n-r}y^{n-R}X^r Y^R. \tag{8}$$

Here the summations in a, A are taken over all zeros of f, F; in $\prod'$ the zero factors $f(a)$, $F(A)$ are to be replaced by $f'(a)$, $F'(A)$; and X, Y are arbitrary parameters independent of x, y.

* Chaundy, loc. cit., 423 (48).

To show that (8) is actually a solution of (6) pick out an element of the four subsidiary summations with fixed a, A, r, R, namely

$$X^{r-a}Y^{R-A}\sum^{n=\infty}\frac{\prod_{t=0}^{n-1}g(a-r+t)G(A-R+t)}{\prod_{t=0}^{n}{}'f(a-r+t)F(A-R+t)}x^{n-r+a}y^{n-R+A}. \qquad (9)$$

The summation begins at $n = \max(r, R)$. It is sufficient to suppose $r \geqslant R$, setting $r \equiv R+k$, $n \equiv r+n'$. Then we may rewrite (9), after removing a factor independent of x, y, as

$$\sum_{n'=0}^{\infty}\frac{\prod_{t=0}^{n'-1}g(a+t)G(A+k+t)}{\prod_{t=1}^{n'}f(a+t)F(A+k+t)}x^{a+n'}y^{A+k+n'},$$

which is just (7). Thus (8) is a solution of (6) involving the two arbitrary parameters X, Y.

We now consider its value when $x = X$ or, more generally, the value of $\delta^m U$ where m is a positive integer or zero. If we perform the differentiation δ^m and the substitution $x = X$ and retain, in the summand of (8), only such factors as contain a, we get

$$\alpha_r \equiv (a+n-r)^m\prod_{t=0}^{n-1}g(a-r+t)\Big/\prod_{t=0}^{n}{}'\,(f(a-r+t),$$

which is the coefficient of $(\tau-a+r)^{-1}$ in the expansion, in partial fractions, of

$$(\tau+n)^m\prod_{t=0}^{n-1}g(\tau+t)\Big/\prod_{t=0}^{n}f(\tau+t).$$

We have supposed f, g to be of respective degrees p, q and their leading coefficients to be unity. Thus

$$\alpha \equiv \sum_{a}\sum_{r=0}^{n}\alpha_r$$

is the coefficient of $\tau^{(n+1)p-1}$ in

$$(\tau+n)^m\prod_{t=0}^{n-1}g(\tau+t),$$

in which the leading term is τ^{m+nq}.

Hence

$$\begin{aligned}
&\alpha = 0 \quad (m < p-1), \quad \text{if } p \geqslant q \text{ and } n \geqslant 0;\\
&\alpha = 0 \quad (m = p-1), \quad \text{if } p > q \text{ and } n > 0;\\
&\alpha = 1 \quad (m = p-1), \quad \text{if } p = q \text{ or } n = 0. \qquad (10)
\end{aligned}$$

Hence, returning to the full summand in U, we see that on $x = X$

(i) $$\delta^m U = 0 \qquad (m < p-1); \tag{11}$$

(ii) if $p > q$,

$$\delta^{p-1} U = \sum_A \left(\frac{y}{Y}\right)^A \Big/ F'(A), \tag{12}$$

all the terms after the first disappearing in the infinite summation: but every A is a zero of F and we get at once from (12) that

$$[F(\delta')\delta^{p-1}U]_{x=X} = 0 \qquad (p > q); \tag{13}$$

(iii) if $p = q$,

$$\delta^{p-1}U = \sum_{n=0}^{\infty} \sum_A \left(\frac{y}{Y}\right)^A \sum_{R=0}^{n} \frac{\prod_{t=0}^{n-1} G(A-R+t)}{\prod_{t=0}^{n}{}' F(A-R+t)} X^n y^{n-R} Y^R,$$

$$= \sum_{n=0}^{\infty} \sum_A \left(\frac{Xy}{XY}\right)^A \sum_{R=0}^{n} \frac{\prod_{t=0}^{n-1} G(A-R+t)}{\prod_{t=0}^{n}{}' F(A-R+t)} (Xy)^{n-R}(XY)^R. \tag{14}$$

Now this is a series exactly similar to (8) but in fewer variables: x, X are absent, and y, Y are replaced by Xy, XY. But we have shown that (8) is a solution of (6), and so, similarly, (14) is a solution of

$$\{F(\delta') - Xy\, G(\delta')\}Z_0 = 0.$$

In other words, we have

$$[\{F(\delta') - xyG(\delta')\}\delta^{p-1}U]_{x=X} = 0 \qquad (p = q). \tag{15}$$

There are evidently analogous results regarding the behaviour of U and its derivatives on $y = Y$.

Finally, at the intersection of $x = X$, $y = Y$, we have from (11), if $p > q$, $$[\delta^{p-1}\delta'^{P-1}U]_{(X,Y)} = \sum_A A^{P-1}/F'(A) = 1,$$

as we see by considering the expansion in partial fractions of $t^{P-1}/F(t)$.

More generally, from symmetry in δ, δ',

$$[\delta^{p-1}\delta'^{P-1}U]_{(X,Y)} = 1 \qquad (p > q \quad \text{or} \quad P > Q). \tag{16}$$

Now, if both $p = q$, $P = Q$, we have by (14),

$$[\delta^{p-1}\delta'^{P-1}U]_{(X,Y)} = \sum_{n=0}^{\infty} \sum_A \sum_{R=0}^{n} (A+n-R)^{P-1} \frac{\prod_{t=0}^{n-1} G(A-R+t)}{\prod_{t=0}^{n}{}' F(A-R+t)} (XY)^n.$$

But, in analogy with (10), we have for all n

$$\sum_{A}\sum_{R=0}^{n}(A+n-R)^{P-1}\frac{\prod_{t=0}^{n-1}G(A-R+t)}{\prod_{t=0}^{n}{}'F(A-R+t)}=1.$$

Thus
$$[\delta^{p-1}\delta'^{P-1}U]_{(X,Y)}=\sum_{n=0}^{\infty}(XY)^n=(1-XY)^{-1}, \tag{17}$$

provided that $|XY|<1$.

3. Relation with the adjoint

In forming the adjoint of (6) I take δ,δ' as the fundamental operators instead of $\partial/\partial x$, $\partial/\partial y$. The adjoint equation is then

$$\Delta' U\equiv\{f(-\delta)F(-\delta)-g(-\delta)G(-\delta')xy\}U=0,$$

i.e.
$$f(-\delta)F(-\delta)U=xy\,g(-\delta-1)G(-\delta'-1)U. \tag{18}$$

So far, $U(x,y;X,Y)$ has been discussed as a solution of the original equation (6), i.e. of $\Delta Z=0$. We shall now see that, if we interchange the pairs of arguments, $U(X,Y;x,y)$ in variables (x,y) is a solution of the adjoint equation (18). If in (8) we make this interchange and at the same time reverse the orders of the summations in r,R and of the products in t, we get

$$U(X,Y;x,y)$$
$$=\sum_{n=0}^{\infty}\sum_{a}\sum_{A}\left(\frac{x}{X}\right)^{-a}\left(\frac{y}{Y}\right)^{-A}\sum_{r=0}^{n}\sum_{R=0}^{n}\frac{\prod_{t=0}^{n-1}g(a-1+r-t)G(A-1+R-t)}{\prod_{t=0}^{n}{}'f(a+r-t)F(A+R-t)}\times$$
$$\times X^rY^Rx^{n-r}y^{n-R}$$
$$=\sum_{n=0}^{\infty}\sum_{a}\sum_{A}\left(\frac{x}{X}\right)^{-a}\left(\frac{y}{Y}\right)^{-A}\sum_{r=0}^{n}\sum_{R=0}^{n}\frac{\prod_{t=0}^{n-1}\bar{g}(-a-r+t)\bar{G}(-A-R+t)}{\prod_{t=0}^{n}{}'\bar{f}(-a-r+t)\bar{F}(-A-R+t)}\times$$
$$\times x^{n-r}y^{n-R}X^rY^R,$$

if we write $f(-z)\equiv\bar{f}(z)$, $g(-1-z)\equiv\bar{g}(z)$, and so for F,G.

Thus $U(X,Y;x,y)$ stands to the adjoint equation

$$\bar{f}(\delta)\bar{F}(\delta')Z=xy\,\bar{g}(\delta)\bar{G}(\delta')Z$$

in the same relation as $U(x,y;X,Y)$ stands to the primitive equation

$$f(\delta)g(\delta')Z=xy\,g(\delta)G(\delta')Z.$$

I write henceforth $U\equiv U(X,Y;x,y)$. Then U is a solution of the

adjoint equation (18); and the properties of $U(x,y;\ X,Y)$ given in (11), (13), (15)–(17) of the preceding section lead to the following properties of U on $x = X$:

$$\delta^m U = 0 \qquad (m < p-1), \tag{19}$$

$$F(-\delta')\delta^{p-1}U = 0 \qquad (p > q), \tag{20}$$

$$\{F(-\delta')-xyG(-\delta'-1)\}\delta^{p-1}U = 0 \qquad (p = q). \tag{21}$$

There are analogous properties on $y = Y$; and finally, at the intersection $x = X$, $y = Y$,

$$(-\delta)^{p-1}(-\delta')^{P-1}U = 1 \qquad (p > q \quad \text{or} \quad P > Q), \tag{22}$$

or

$$= (1-XY)^{-1} \qquad (p = q;\ P = Q). \tag{23}$$

To exhibit concisely the characteristic adjoint identity it is convenient to denote by $f_r(z)$ the quotient when $f(z)$ is divided by z^r $(r = 0, 1, \ldots, p)$, so that, in particular, $f_0(z) \equiv f(z)$ and $f_p(z) \equiv 1$; and similarly for g, F, G. Thus $f_r(z)-zf_{r+1}(z)$ is the coefficient of z^r in $f(z)$, and we deduce at once the identity

$$\delta\sum_{r=0}^{p-1}(-\delta)^r V\,.\,f_{r+1}(\delta)W = V\,.\,f(\delta)W-f(-\delta)V\,.\,W, \tag{24}$$

where V, W are any suitable operands, and the full point $(.)$ delimits the action of operators affecting V. Write also

$$\begin{aligned}\Phi_{(x)y} &\equiv \sum_{r=0}^{p-1}(-\delta)^r U\,.\,f_{r+1}(\delta)F(\delta')Z-\sum_{s=0}^{q-1}(-\delta)^s xyU\,.\,g_{s+1}(\delta)G(\delta')Z \\ \Phi_{x(y)} &\equiv \sum_{R=0}^{P-1}f(-\delta)(-\delta')^R U\,.\,F_{R+1}(\delta')Z-\sum_{S=0}^{Q-1}g(-\delta)(-\delta')^S xyU\,.\,G_{S+1}(\delta')Z\end{aligned} \tag{25}$$

with two similar formulae obtained by interchange of x, y; and further

$$\begin{aligned}\Phi_{(xy)} \equiv \sum_{r=0}^{p-1}\sum_{R=0}^{P-1}(-\delta)^r(-\delta')^R U\,.\,f_{r+1}(\delta)F_{R+1}(\delta')Z- \\ -\sum_{s=0}^{q-1}\sum_{S=0}^{Q-1}(-\delta)^s(-\delta')^S xyU\,.\,g_{s+1}(\delta)G_{S+1}(\delta')Z,\end{aligned} \tag{26}$$

where evidently $\Phi_{(xy)} = \Phi_{(yx)}$.

Application of (24) to (26) gives in terms of (25)

$$\delta\Phi_{(xy)} = \Phi_{(y)x}-\Phi_{x(y)}; \tag{27}$$

and by use of (24) and its analogue in δ' we get again

$$\delta\Phi_{(x)y}+\delta'\Phi_{x(y)} = U\Delta Z-Z\Delta' U = 0, \tag{28}$$

if U, Z satisfy respectively the adjoint equation and its primitive.

This last is the characteristic adjoint identity. I prefer, however, to employ it in the more symmetrical form

$$\delta\delta'\Phi_{(xy)}-\delta\Phi_{(x)y}-\delta'\Phi_{(y)x}=0, \tag{29}$$

which is readily deducible from (27), (28).

Now consider the value of $\Phi_{y(x)}$ on $x=X$. By (19) it reduces to

$$ZF(-\delta')(-\delta)^{p-1}U \qquad (p>q),$$

or $$Z\{F(-\delta')-xyG(-\delta'-1)\}(-\delta)^{p-1}U \qquad (p=q).$$

In either case then, by (20), (21), $\Phi_{y(x)}$ vanishes on $x=X$, and so, from the analogue of (27),

$$\Phi_{(x)y}=\delta'\Phi_{(xy)} \quad \text{on} \quad x=X. \tag{30}$$

Again, at $x=X$, $y=Y$, by (19) and its analogue,

$$\Phi_{(xy)}=Z(-\delta)^{p-1}(-\delta')^{P-1}U \qquad (p>q \text{ or } P>Q),$$

or $$=Z(1-XY)(-\delta)^{p-1}(-\delta')^{P-1}U \quad (p=q;\ P=Q).$$

Hence in either case, by (22), (23),

$$\Phi_{(xy)}=Z \quad \text{at} \quad x=X, \quad y=Y. \tag{31}$$

4. The general solution

To obtain the general solution I replace the appeal to Stokes's theorem by a modified form of procedure, although the underlying analysis is, of course, essentially the same. If we integrate the adjoint identity (29) over the rectangle with opposite vertices at (X, Y) and (ξ, η), where (ξ, η) is some suitable origin, we get

$$\int_\xi^X\int_\eta^Y \frac{dxdy}{xy}\delta\delta'\Phi_{(xy)} = \int_\xi^X\int_\eta^Y \frac{dxdy}{xy}\delta\Phi_{(x)y}+\int_\xi^X\int_\eta^Y \frac{dxdy}{xy}\delta'\Phi_{(y)x}$$

$$=\int_\eta^Y \frac{dy}{y}[\Phi_{(x)y}]_\xi^X+\int_\xi^X \frac{dx}{x}[\Phi_{(y)x}]_\eta^Y.$$

Thus, by (30) and its analogue,

$$\int_\xi^X \frac{dx}{x}[\Phi_{(y)x}]_{y=\eta}+\int_\eta^Y \frac{dy}{y}[\Phi_{(x)y}]_{x=\xi}$$

$$=\int_\xi^X \frac{dx}{x}[\delta\Phi_{(xy)}]_{y=Y}+\int_\eta^Y \frac{dy}{y}[\delta'\Phi_{(xy)}]_{x=X}-\int_\xi^X\int_\eta^Y \frac{dxdy}{xy}\delta\delta'\Phi_{(xy)}$$

$$=[\Phi_{(xy)}]_{(\xi,\eta)}^{(X,Y)}, \quad \text{on performing the integrations,}$$

$$=Z(X, Y)-[\Phi_{(xy)}]_{\substack{x=\xi\\ y=\eta}}, \quad \text{by (31).}$$

Hence, finally,

$$Z(X,Y)=\int_{\xi}^{X}\frac{dx}{x}[\Phi_{(y)x}]_{y=\eta}+\int_{\eta}^{Y}\frac{dy}{y}[\Phi_{(x)y}]_{x=\xi}+[\Phi_{(xy)}]_{\substack{x=\xi\\y=\eta}}. \quad (32)$$

If we substitute for the Φ's from (25), (26), we have here the general solution of (6). It is, however, slightly better to rewrite

$$\Phi_{(x)y}=\sum_{r=0}^{p-1}f_{r+1}(-\delta)U\,.\,\delta^{r}F(\delta')Z-xy\sum_{s=0}^{q-1}g_{s+1}(-\delta)U\,.\,\delta^{s}G(\delta')Z, \quad (33)$$

$$\Phi_{(xy)}=\sum_{r=0}^{p-1}\sum_{R=0}^{P-1}f_{r+1}(-\delta)F_{R+1}(-\delta')U\,.\,\delta^{r}\delta'^{R}Z-$$
$$-xy\sum_{s=0}^{q-1}\sum_{S=0}^{Q-1}g_{s+1}(-\delta-1)G_{S+1}(-\delta'-1)U\,.\,\delta^{s}\delta'^{S}Z, \quad (34)$$

with an analogous expression for $\Phi_{(y)x}$.

Thus, at full length, (32) may be written

$$Z(X,Y)=\int_{\xi}^{X}\Big\{\sum_{R=0}^{P-1}F_{R+1}(-\delta'_{\eta})U(X,Y;x,\eta)\,.f(\delta)\delta'^{R}_{\eta}Z(x,\eta)-$$
$$-x\eta\sum_{S=0}^{Q-1}G_{S+1}(-\delta'_{\eta}-1)U(X,Y;x,\eta)\,.g(\delta)\delta'^{S}_{\eta}Z(x,\eta)\Big\}\frac{dx}{x}+$$
$$+\int_{\eta}^{Y}\Big\{\sum_{r=0}^{p-1}f_{r+1}(-\delta_{\xi})U(X,Y;\xi,y)\,.F(\delta')\delta^{r}_{\xi}Z(\xi,y)-$$
$$-\xi y\sum_{s=0}^{q-1}g_{s+1}(-\delta_{\xi}-1)U(X,Y;\xi,y)\,.G(\delta')\delta^{s}_{\xi}Z(\xi,y)\Big\}\frac{dy}{y}+$$
$$+\sum_{r=0}^{p-1}\sum_{R=0}^{P-1}f_{r+1}(-\delta_{\xi})F_{R+1}(-\delta'_{\eta})U(X,Y;\xi,\eta)\,.\delta^{r}_{\xi}\delta'^{R}_{\eta}Z(\xi,\eta)-$$
$$-\xi\eta\sum_{s=0}^{q-1}\sum_{S=0}^{Q-1}g_{s+1}(-\delta_{\xi}-1)G_{s+1}(-\delta'_{\eta}-1)U(X,Y;\xi,\eta)\,.\delta^{s}_{\xi}\delta'^{S}_{\eta}Z(\xi,\eta). \quad (35)$$

Here (ξ,η) is some conveniently-chosen origin, $\delta_{\xi}, \delta'_{\eta}$ denote, of course, $\partial/\partial\xi$, $\partial/\partial\eta$, and X, Y are now current coordinates. With these conventions we may say that

in coordinates (X, Y), *the formula* (35) *gives the general solution of the hypergeometric partial differential equation of hyperbolic type*

$$f(\delta)F(\delta')Z=xy\,g(\delta)G(\delta')Z \qquad (p\geqslant q;P\geqslant Q).$$

If we are looking for the general solution in terms of arbitrary functions, then (33) gives it in terms of the $p+P$ distinct arbitrary functions

$$\delta'^{R}_{\eta}Z(x,\eta) \quad (R=0,1,\ldots,P-1), \qquad \delta^{r}_{\xi}Z(\xi,y) \quad (r=0,1,\ldots,p-1),$$

equal in number, as we should expect, to the order $p+P$ of the differential equation. The solution also involves the pP arbitrary constants

$$\delta_\xi^r \delta_\eta'^R Z(\xi, \eta) \qquad (r = 0, 1, \ldots, p-1;\ R = 0, 1, \ldots, P-1),$$

which may be distinct (or not) from special values of the arbitrary functions.

We may also regard (33) as giving the solution of 'Cauchy's problem', the data being borne by the characteristics $x = \xi$, $y = \eta$: in this case the data are the values on $x = \xi$ of

$$\delta^r F(\delta')Z \quad (r = 0, 1, \ldots, p-1), \qquad \delta^s G(\delta')Z \quad (s = 0, 1, \ldots, q-1)$$

and the values on $y = \eta$ of

$$\delta'^R f(\delta)Z \quad (R = 0, 1, \ldots, P-1), \qquad \delta'^S g(\delta)Z \quad (S = 0, 1, \ldots, Q-1).$$

More generally, if the data are borne by a simple curve C, we now take (X, η), (ξ, Y) to be the points in which this curve is cut by the characteristics $x = X$, $y = Y$. We shall then obtain the solution of Cauchy's problem for the given curve C in the form

$$Z(X, Y) = \tfrac{1}{2}[\Phi_{(xy)}]_{(X,\eta)} + \tfrac{1}{2}[\Phi_{(xy)}]_{(\xi, Y)} - \\ - \frac{1}{2}\int_{(X,\eta)}^{(\xi,Y)} \left\{(\Phi_{x(y)} + \Phi_{(y)x})\frac{dx}{x} - (\Phi_{(x)y} + \Phi_{y(x)})\frac{dy}{y}\right\}, \quad (36)$$

where the integration is taken along the base curve C, and the Φ's have the meanings assigned in (25), (26), or equivalently in (33), (34)

To prove this we write the adjoint identity (28) in the symmetrical form

$$\delta(\Phi_{(x)y} + \Phi_{y(x)}) + \delta'(\Phi_{x(y)} + \Phi_{(y)x}) = 0$$

and apply Stokes's theorem to the function on the left over the area bounded by the characteristics $x = X$, $y = Y$ and the basic curve C. Since the integrand is everywhere zero in the area, the integral along the boundary also vanishes, and we get

$$\int_{(X,\eta)}^{(\xi,Y)} \left\{(\Phi_{x(y)} + \Phi_{(y)x})\frac{dx}{x} - (\Phi_{(x)y} + \Phi_{y(x)})\frac{dy}{y}\right\} \\ = \int_\eta^Y \frac{dy}{y}[\Phi_{x(y)} + \Phi_{(y)x}]_{x=X} + \int_\xi^X \frac{dx}{x}[\Phi_{(x)y} + \Phi_{y(x)}]_{y=Y} \\ = \int_\eta^Y \frac{dy}{y}[\delta'\Phi_{(xy)}]_{x=X} + \int_\xi^X \frac{dx}{x}[\delta\Phi_{(xy)}]_{y=Y},$$

by (27), (30) and their analogues

$$= 2[\Phi_{(xy)}]_{(X,Y)} - [\Phi_{(xy)}]_{(X,\eta)} - [\Phi_{(xy)}]_{(\xi,Y)},$$

which proves the theorem.

5. Range of the solution

We now consider how much is covered by the 'general' solution just obtained. We begin by examining the region of convergence of the fundamental solution (8), which we may write in the form

$$U(x,y;X,Y) = \sum_a \sum_b \sum_{n=0}^{\infty} u_n v_n, \tag{37}$$

where

$$u_n \equiv \sum_{r=0}^{n} \frac{\prod_{t=0}^{n-1} g(a+t-r)}{\prod_{t=0}^{n}{}' f(a+t-r)} x^{n-r} X^r, \tag{38}$$

and v_n is the analogous expression in y, Y. Splitting f, g into their linear factors, say

$$f(z) \equiv \prod_{i=1}^{p} (z-a_i), \qquad g(z) \equiv \sum_{j=1}^{q} (z-b_j),$$

we have

$$u_n = \sum_{r=0}^{n} \frac{\prod_{j=0}^{q} \prod_{t=0}^{n-1} (a+t-r-b_j) x^{n-r} X^r}{(-)^r r!\,(n-r)! \prod_{i=0}^{p}{}' \prod_{t=0}^{q} (a+t-r-a_i)}, \tag{39}$$

where $\prod'$ in the denominator omits factors in which $a_i = a$, these factors combining to give the numerical factor $(-)^r r!\,(n-r)!$.

Now we have expressly excluded the case in which any $a-a_i$ is an integer or zero; so too in the numerator we can exclude any case in which $a-b_j$ is an integer or zero, for this gives ultimately a zero factor and so a vanishing product. We may therefore reduce consideration of u_n to consideration of the types of product

$$\prod_{t=0}^{n-1} (\lambda+t-r) \quad \text{or} \quad \prod_{t=0}^{n} (\lambda+t-r) \qquad (0 \leqslant r \leqslant n)$$

where λ is not an integer (or zero). Now

$$\prod_{t=0}^{n-1} (\lambda+t-r) = (-)^r r!\,(n-r)! \prod_{s=1}^{r} \left(1-\frac{\lambda}{s}\right) \prod_{s=1}^{n-r} \left(1+\frac{\lambda-1}{n-s}\right), \tag{40}$$

and, by Euler's definition of the gamma-function*

$$(N+1)^{-\mu}\prod_{s=1}^{N}\left(1+\frac{\mu}{s}\right)\to 1/\mu\Gamma(\mu)\qquad(N\to\infty).$$

The expansion on the left is therefore bounded for fixed μ not an integer (or zero), and we may write

$$\prod_{s=1}^{N}\left(1+\frac{\mu}{s}\right)\sim C_\mu(N+1)^\mu,$$

where C_μ depends on μ only. Hence, from (40), we may write

$$\prod_{t=0}^{n-1}|\lambda+t-r|\sim C_\lambda r!\,(n-r)!\,(r+1)^{-\lambda}(n-r+1)^{\lambda-1},$$

and, by the insertion of an additional factor,

$$\prod_{t=0}^{n}|\lambda+t-r|\sim C'_\lambda r!\,(n-r)!\,(r+1)^{-\lambda}(n-r+1)^{\lambda}.$$

Putting these results in (39) we get

$$|u_n|<C\sum_{r=0}^{n}\{r!\,(n-r)!\}^{q-p}(r+1)^{-h}(n-r+1)^{h-q}\bar{x}^n,$$

where $\qquad \bar{x}\equiv\max(|x|,|X|),\qquad h\equiv\sum(a_i-b_j),$

and C depends only on the constants of the equation. Now $r!\,(n-r)!>([\frac{1}{2}n]!)^2$, and $(n-r+1)/(r+1)$, being monotonic in r, lies between its extreme values $n+1$, $(n+1)^{-1}$. Thus, taking a wide inequality, we have

$$|u_n|<C([\tfrac{1}{2}n]!)^{-2(p-q)}(n+1)^{|h|+1}\bar{x}^n,$$

with, of course, a similar inequality for v_n. Hence, if $p>q$ or $P>Q$, the series U converges for all x, y, X, Y.

But, if $p=q$ and $P=Q$, the preceding inequality gives convergence only if $\bar{x}\bar{y}<1$, where $\bar{y}\equiv\max(|y|,|Y|)$. Closer examination and preciser conditions might give convergence right on the boundary $\bar{x}\bar{y}=1$, as with the ordinary hypergeometric series, but I shall not now stop to consider this limiting case. If, then, we draw the pair of conjugate hyperbolas $xy=\pm1$, there is certainly convergence if the four points (x,y), (X,Y), (X,y), (x,Y) be within the cruciform region bounded by the two hyperbolas, i.e. if the rectangle having sides parallel to the axes and opposite vertices at (x,y), (X,Y) lies entirely within the cruciform area.

* See, for instance, Whittaker and Watson, *Modern Analysis* (Cambridge, 1920), 237, § 12.11.

If now in the general solution (35) we take (ξ, η) at the origin of coordinates, these conditions of convergence are satisfied so long as $|XY| < 1$, a condition we have already presumed in (17). Thus (35) is valid throughout the interior of the cruciform area, provided only that $U(X, Y; x, y)$ exists at $x = 0$, $y = 0$.

Now $U(X, Y; x, y)$ is expressed as the sum of a finite number of ascending series with leading terms $x^{-a}y^{-A}$, and so it exists on the axes of coordinates, if every a, A, i.e. every zero of f, F, is negative. But the substitution $Z = x^h y^k Z'$ subtracts h, k from every zero of f, F respectively and so enables us to smooth out any difficulties along the axes. Actually the difficulty is a formal one: the possible singularities of U on the axes are at worst 'accidental singularities', being poles of orders (a, b), etc., and it will be found that these are matched by corresponding zeros in Z. The point requires more elucidation than can be afforded here, but we are at least justified in saying that (35) provides a general solution valid throughout the interior of $|XY| < 1$. The analysis of §§ 2–4, as examination will show, imposes no condition on Z other than that of differentiability up to orders implied in the differential equation itself, i.e. p in x and q in y: we do not need to suppose, for instance, that Z is analytic or even that derivatives exist beyond those just stated; but, in the light of what has been said above, we must probably add conditions regarding the poles and zeros of Z along the axes.

It is, of course, to be expected that the hyperbola $xy = 1$ should be a singular curve in the case $p = q$, $P = Q$, since then $1 - xy$ is the coefficient of the leading term in the differential equation, the conjugate hyperbola being derived, as usual, by reflexion in the axes.

In the quadripartite region $|XY| > 1$ outside these bounding hyperbolas we write the differential equation as

$$g(\delta)G(\delta')Z = (xy)^{-1}f(\delta)F(\delta')Z$$

and repeat in descending series the analysis that has so far been based solely on ascending series. There is strict analogy with the ascending and descending solutions of ordinary hypergeometric equations.* This applies, of course, only to the case in which $p = q$, $P = Q$. Otherwise, with $p > q$ or $P > Q$, the ascending series are integral functions, and the solution is valid throughout the (X, Y)-plane.

* See, for instance, Chaundy, loc. cit., 423.

We may sum up our conclusions by saying that

(i) *if $p > q$ or $P > Q$, the solution* (35), *for arbitrary* (ξ, η), *includes every solution* $Z(X, Y)$ *of* (6) *that is differentiable up to an order implied by the differential equation itself, subject to possible restrictions on its behaviour along the axes: and it will define this solution throughout the plane of* (X, Y);

(ii) *if $p = q$, $P = Q$, the solution* (35), *with $\xi = 0 = \eta$, includes every solution* $Z(X, Y)$ *of* (6) *that exists in the region $|XY| < 1$ and is differentiable up to an order implied by the differential equation itself, subject to possible restrictions on its behaviour along the axes: and there is a corresponding general solution, based on descending series, appropriate to the region $|XY| > 1$.*

6. Extension to many variables

The analysis extends without difficulty to the hypergeometric partial differential equation in n independent variables

$$\prod_{r=1}^{n} f_r(\delta_r) Z = \prod_{r=1}^{n} x_r g_r(\delta_r) Z,$$

where $\delta_r \equiv x_r \partial/\partial x_r$. It will be sufficient to note, without proof, the crucial points in the analysis for the case $n = 3$, i.e. for the equation

$$f_1(\delta_1) f_2(\delta_2) f_3(\delta_3) Z = x_1 x_2 x_3 g_1(\delta_1) g_2(\delta_2) g_3(\delta_3) Z, \tag{41}$$

where f_r, g_r have orders p_r, q_r $(p_r \geqslant q_r)$ and leading coefficients unity.

The fundamental solution is

$$\begin{aligned} &U(x_1, x_2, x_3;\ X_1, X_2, X_3) \\ &\equiv \sum_{n=0}^{\infty} \Bigg\{ \sum_{a_1, a_2, a_3} \left(\frac{x_1}{X_1}\right)^{a_1} \left(\frac{x_2}{X_2}\right)^{a_2} \left(\frac{x_3}{X_3}\right)^{a_3} \times \\ &\quad \times \sum_{r_1=0}^{n} \sum_{r_2=0}^{n} \sum_{r_3=0}^{n} \frac{\prod_{t=0}^{n-1} g_1(a_1 - r_1 + t) g_2(a_2 - r_2 + t) g_3(a_3 - r_3 + t)}{\prod_{t=0}^{n}{}' f_1(a_1 - r_1 + t) f_2(a_2 - r_2 + t) f_3(a_3 - r_3 + t)} \times \\ &\qquad \times x_1^{n-r_1} x_2^{n-r_2} x_3^{n-r_3} X_1^{r_1} X_2^{r_2} X_3^{r_3} \Bigg\}, \end{aligned} \tag{42}$$

where a_r is a typical zero of f_r. From this, by interchange of x_r, X_r, we derive the solution $U \equiv U(X_1, X_2, X_3;\ x_1, x_2, x_3)$ of the adjoint equation $\Delta' U = 0$. On the 'special' characteristics U has the following properties:

on the 'plane' $x_1 = X_1$,

$$\delta_1^m U = 0 \ (m < p_1 - 1), \qquad f_2(-\delta_2)\delta_1^{p_1-1}U = 0, \text{ etc.} \quad (p_1 > q_1),$$

$$\{f_2(-\delta_2)f_3(-\delta_3) - g_2(-\delta_2)g_3(-\delta_3)x_1x_2x_3\}\delta_1^{p_1-1}U = 0 \quad (p_1 = q_1);$$

along the 'line' $x_1 = X_1, x_2 = X_2$,

$$\{f_3(-\delta_3) - g_3(-\delta_3)x_1x_2x_3\}\delta_1^{p_1-1}\delta_2^{p_2-1}U = 0 \quad (p_1 = q_1;\ p_2 = q_2);$$

at the 'point' $x_1 = X_1, x_2 = X_2, x_3 = X_3$,

$$(-\delta_1)^{p_1-1}(-\delta_2)^{p_2-1}(-\delta_3)^{p_3-1}U = 1 \qquad (\text{some } p_r > q_r),$$

or

$$= (1 - X_1X_2X_3)^{-1} \quad (\text{every } p_r = q_r).$$

The generalization of the symbols Φ may be indicated by defining for *many* variables

$$\Phi_{\alpha\ldots(\beta\ldots)\gamma\ldots} \equiv \sum_{r_\beta=0}^{p_\beta-1} \ldots f_\alpha(-\delta_\alpha)\ldots(-\delta_\beta)^{r_\beta}\ldots U \,.\, f_\beta^{(r_\beta+1)}(\delta_\beta)\ldots f_\gamma(\delta_\gamma)\ldots Z,$$

where $\alpha\ldots, \beta\ldots, \gamma\ldots$ denote any suffixes $1, 2, 3, \ldots$ in any order, and $f^{(r)}$ is now written in place of f_r. Then, by lemma (24),

$$\delta_\theta \Phi_{\alpha\ldots(\theta\beta\ldots)\gamma\ldots} = \Phi_{\alpha\ldots(\beta\ldots)\theta\gamma\ldots} - \Phi_{\theta\alpha\ldots(\beta\ldots)\gamma\ldots}.$$

Returning to three independent variables we have on the special characteristics

$$\Phi_{\alpha\beta(\gamma)} = 0 \quad \text{on } x_\gamma = X_\gamma, \qquad \Phi_{\alpha(\beta\gamma)} = 0 \quad \text{along } x_\beta = X_\beta,\ x_\gamma = X_\gamma,$$

$$\Phi_{(123)} = Z \quad \text{at } x_1 = X_1,\ x_2 = X_2,\ x_3 = X_3.$$

The adjoint identity is now

$$\delta_\alpha \Phi_{(\alpha)\beta\gamma} + \delta_\beta \Phi_{\alpha(\beta)\gamma} + \delta_\gamma \Phi_{\alpha\beta(\gamma)} = 0,$$

or, as I prefer to write it,

$$\sum \delta_1 \Phi_{(1)23} - \sum \delta_2\delta_3 \Phi_{(23)1} + \delta_1\delta_2\delta_3 \Phi_{(123)} = 0,$$

both summations being over the suffixes 1, 2, 3.

Apply to this the triple integration

$$\int_{\xi_1}^{X_1}\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3} (\ )\frac{dx_1\,dx_2\,dx_3}{x_1x_2x_3},$$

perform the integrations offered by δ_1, etc. in the first summation, and in the second write, as we may,

$$\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3} \frac{dx_2\,dx_3}{x_2x_3}\delta_2\delta_3\Phi = \int_{\xi_3}^{X_3} \frac{dx_2}{x_2}[\delta_2\Phi]_{x_3=X_3} + \int_{\xi_3}^{X_3} \frac{dx_3}{x_3}[\delta_3\Phi]_{x_2=X_2} + [\Phi]_{(X_2,X_3)}^{(\xi_2,\xi_3)},$$

and so forth.

We then get, after rearrangement,

$$\sum\int_{\xi_3}^{X_3}\int_{\xi_2}^{X_2}\frac{dx_2\,dx_3}{x_2x_3}[\Phi_{(1)23}]_{x_1=\xi_1}+\sum\int_{\xi_1}^{X_1}\frac{dx_1}{x_1}[\Phi_{(23)1}]_{\substack{x_2=\xi_2,\\x_3=\xi_3}}$$

$$=\int_{\xi_1}^{X_1}\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3}\frac{dx_1\,dx_2\,dx_3}{x_1x_2x_3}\delta_1\delta_2\delta_3\Phi_{(123)}+$$

$$+\sum\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3}\frac{dx_2\,dx_3}{x_2x_3}[\Phi_{(1)23}-\delta_2\Phi_{(12)3}-\delta_3\Phi_{(13)2}]_{x_1=X_1}+$$

$$+\sum\int_{\xi_1}^{X_1}\frac{dx_1}{x_1}[\delta_1\Phi_{(23)1}]_{\substack{x_2=X_2,\\x_3=X_3}} \tag{43}$$

But

$$\Phi_{(1)23}-\delta_2\Phi_{(12)3}-\delta_3\Phi_{(13)2}=\delta_2\delta_3\Phi_{(123)}-\Phi_{23(1)}$$
$$=\delta_2\delta_3\Phi_{(123)}\quad\text{on}\quad x_1=X_1;$$

and

$$\Phi_{(23)1}=\delta_1\Phi_{(123)}+\Phi_{1(23)}=\delta_1\Phi_{(123)}\quad\text{along } x_2=X_2,\ x_3=X_3.$$

Thus we can write the right-hand side of (43) as

$$\int_{\xi_1}^{X_1}\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3}\frac{dx_1\,dx_2\,dx_3}{x_1x_2x_3}\delta_1\delta_2\delta_3\Phi_{(123)}-\sum\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3}\frac{dx_2\,dx_3}{x_2x_3}[\delta_2\delta_3\Phi_{(123)}]_{x_1=X_1}+$$

$$+\sum\int_{\xi_1}^{X_1}\frac{dx_1}{x_1}[\delta_1\Phi_{(123)}]_{\substack{x_2=X_2,\\x_3=X_3}}$$

$$=[\Phi_{(123)}]_{(\xi_1,\xi_2,\xi_3)}^{(X_1,X_2,X_3)},$$

on integration and cancellation. Hence, finally, we have

$$Z(X_1,X_2,X_3)$$

$$=\sum_{1,2,3}\int_{\xi_2}^{X_2}\int_{\xi_3}^{X_3}\frac{dx_2\,dx_3}{x_2x_3}[\Phi_{(1)23}]_{x_1=\xi_1}+\sum_{1,2,3}\int_{\xi_1}^{X_1}\frac{dx_1}{x_1}[\Phi_{(23)1}]_{\substack{x_2=\xi_2,\\x_3=\xi_3}}+[\Phi_{(123)}]_{\substack{x_1=\xi_1,\\x_2=\xi_2,\\x_3=\xi_3}}. \tag{44}$$

For the solution of Cauchy's problem when the data are borne by a surface S we integrate some form of the adjoint identity throughout the volume bounded by S and the three characteristics

$$x_r=X_r\ (r=1,2,3).$$

The discussion of the convergence of U and the range of validity of the solution in many variables introduces no new difficulties.

NOTES ON THE THEORY OF SERIES (XIX): A PROBLEM CONCERNING MAJORANTS OF FOURIER SERIES

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1. Statement of the problem

1.1. SUPPOSE that $f(\theta)$ is a complex and integrable function of the real variable θ, with period 2π, and that

$$c_m = c_m(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-mi\theta}\,d\theta$$

is its typical complex Fourier constant, so that

$$\mathbf{S}(f) = \sum_{-\infty}^{\infty} c_m e^{mi\theta} \tag{1.1.1}$$

is the Fourier series of $f(\theta)$. We call any formal series

$$\sum_{-\infty}^{\infty} C_m e^{mi\theta} \tag{1.1.2}$$

in which $$C_m \geqslant |c_m|$$

a majorant of (1.1.1). If (1.1.2) is itself a Fourier series, say the Fourier series $\mathbf{S}(F)$ of $F(\theta)$, then we say that F is a majorant of f; and we write

$$F \succ f, \qquad f \prec F, \qquad \mathbf{S}(F) \succ \mathbf{S}(f), \qquad \mathbf{S}(f) \prec \mathbf{S}(F).$$

If $$C_m = |c_m|$$

for every m, we say that F is the exact majorant of f.

1.2. We consider relations of inequality between

$$J_r(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r\,d\theta\right)^{1/r},$$

where $r > 1$, and $J_r(F)$.† There are special cases in which it is easy to establish such relations.

(*a*) If $r = 2$ then

$$J_r(f) = (\sum |c_m|^2)^{\frac{1}{2}} \leqslant (\sum C_m^2)^{\frac{1}{2}} = J_r(F).$$

In this case there is equality when F is the exact majorant.

† If f or F is not L^r, then $J_r(f)$ or $J_r(F)$ is to be interpreted as ∞. This gloss does not affect our results and need not be referred to again.

(*b*) If $r = 2k$, where k is a positive integer, and $J_{2k}(f) < \infty$, then the Fourier series of f^k is $\sum c_m^{(k)} e^{mi\theta}$, where

$$c_m^{(k)} = \sum_{m_1+m_2+\ldots+m_k=m} c_{m_1} c_{m_2} \ldots c_{m_k};$$

and

$$|c_m^{(k)}| \leqslant C_m^{(k)}, \qquad f^k \prec F^k.$$

Hence

$$J_{2k}(f) = J_2^{2/k}(f^k) \leqslant J_2^{2/k}(F^k) = J_{2k}(F) \tag{1.2.1}$$

for any majorant F.

(*c*) Let $r = 1$, $\phi = |f|^{\frac{1}{2}}$, and let Φ be the exact majorant of ϕ. Then $F = \Phi^2$ is a majorant of f, and for this special majorant

$$J_1(F) = J_2^2(\Phi) = J_2^2(\phi) = J_1(f).$$

1.3. In what follows we suppose

$$r > 1,\dagger \qquad 1 < p \leqslant 2, \qquad q > 2.$$

As usual, we write

$$r' = \frac{r}{r-1}, \qquad \frac{1}{r} + \frac{1}{r'} = 1.$$

Here r may be a p or a q; p' is a q and q' a p.

The first suggestion of (1.2.1) is that

$$J_q(f) \leqslant J_q(F) \tag{1.3.1}$$

for every q and every majorant F. This is untrue and, since it is the falsity of (1.3.1) which first reveals the difficulties of our problem, we prove it at once by an example.

If

$$f(z) = 1 + z - az^3,$$

where a is real and positive, then

$$\phi = f^{\frac{1}{2}} = 1 + \tfrac{1}{2}z + \tfrac{1}{8}z^2 - cz^3 + \ldots,$$

where

$$c = \tfrac{1}{16} + \tfrac{1}{2}a,$$

for small z. Hence, if $z = re^{i\theta}$ and r is small,‡

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^3 \, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(z)|^2 \, d\theta = 1 + \tfrac{9}{4}r^2 + \tfrac{9}{64}r^4 + c^2 r^6 + \ldots,$$

which is greater than it would be if a were replaced by $-a$. In other words, if

$$f(\theta) = 1 - re^{i\theta} - ar^3 e^{3i\theta}$$

and F is the exact majorant of f, then

$$J_3(f) > J_3(F)$$

for sufficiently small r; and this contradicts (1.2.1).

† (*c*) of § 1.2 contains all we have to say about the case $r = 1$.

‡ This use of r is temporary and will not cause confusion.

1.4. It remains possible that

$$J_q(f) \leqslant A_q J_q(F), \tag{1.4.1}$$

where A_q is a function of q only (and $A_q = 1$ for $q = 2, 4, 6, \ldots$). Whether this is true or not we cannot say, but it is a quite plausible conjecture; and the conjecture leads naturally to another. In problems of this character there is very usually a 'skew symmetry', resulting in a reversal of the sign of inequality, about the index 2;† and it is natural to suppose that, if (1.4.1) is true, then also

$$J_p(F) \leqslant A_p J_p(f), \tag{1.4.2}$$

not indeed for every majorant F (which is plainly impossible), but at any rate for *some* majorant.

1.5. Our main theorem here (Theorem 1) contains a partial solution of this problem. We cannot prove (1.4.1) or refute it, but we can show that (1.4.1) implies (1.4.2). More precisely, the truth of (1.4.1), for a particular q (and all majorants F), implies that of (1.4.2) for $p = q'$ (and some majorant F). Since (1.4.1) is true at any rate when $q = 2k$, we obtain a proof of (1.4.2), which is not trivial in any case (apart from $p = 2$), for $p = \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots$.

It is instructive to contrast our argument with those used, by other writers and by ourselves, in similar problems such as the Young-Hausdorff problem or M. Riesz's problem of conjugate functions.‡ In all these problems the case $p = q = 2$ is trivial, and the full proof may be divided into three stages. We have to establish two propositions

$$P(p), \qquad Q(q);$$

Q may be (as in Riesz's problem) of the same form as P, but more usually (as here) it is not. We proceed as follows.

(*a*) We prove that $\qquad Q(q) \rightarrow P(q')$.§

This proposition ('reciprocation') reduces the problem to the proof of $Q(q)$.

(*b*) We prove that $\qquad Q(q_1) \,.\, Q(q_2) \rightarrow Q(q)$

for $q_1 \leqslant q \leqslant q_2$. This proposition ('interpolation') reduces the proof

† For example in the Hausdorff inequalities

$$J_{p'}(f) \leqslant (\Sigma|c_m|p)^{1/p}, \quad (\Sigma|c_m|q)^{1/q} \leqslant J_{q'}(f);$$

or in our own inequalities involving

$$J_p(f), (\Sigma|m|+1)^{p-2}|c_m|^p, \ldots, \ldots .$$

‡ The original model is Hausdorff's (Hausdorff, **4**).

§ q' is a p. The symbol $\rightarrow$, when standing between *propositions*, is Hilbert's symbol of implication.

of $Q(q)$ either to a proof of $Q(q)$ in the two extreme cases, $q = 2$ and $q = \infty$,† or to its proof for some particular sequence of values of q, usually the even integers.‡ When the operations or functionals involved in the problem are *linear*, 'interpolation' may be inferred from M. Riesz's theorem concerning the convexity of linear functionals.§

(*c*) We prove, say, $Q(2k)$, and this generally requires some special trick. It is here that the individuality of the problem is likely to declare itself most clearly, the other stages of the proof being usually effected by appeals to general theorems or arguments now of standardized types.

This is the standard position; the position here is abnormal. Here stage (*c*) of the argument is trivial, as we saw in § 1.2. Stage (*b*), on the other hand, collapses altogether. The operations involved here are not linear; F, even the exact majorant, is not a linear functional of f. There is therefore no general theorem to which we can appeal for the proof of 'interpolation', and we are unable to prove it by any special device. Indeed 'interpolation', in the most obvious sense (and the sense in which it would be true if the operations *were* linear), is *false*, since (1.2.2) is true for $q = 2k$ and not for general q.

It is perhaps rather surprising in these circumstances that, in spite of the non-linearity of our operations, we can put through stage (*a*) of the argument and so prove Theorem 1. The arguments generally follow familiar lines, but the non-linearity gives them some curious twists.

2. Proof of the theorem

2.1. Our main theorem‖ is

THEOREM 1. *If p' is a q for which* (1.4.1) *is true, for every f and* EVERY *majorant F, and in particular if*

$$p = \frac{2k}{2k-1}, \qquad p' = 2k \qquad (k = 1, 2, 3, \ldots),$$

then (1.4.2) *is true for every f and* SOME *majorant F.*

We write

$$f_n = \sum_{-n}^{n} c_m e^{mi\theta}.$$

† As in F. Riesz's proof of Hausdorff's theorems; see F. Riesz (**6**).

‡ As in Hausdorff's own proof of his theorems (where the case $q = 2k$ is derived from Young), in one of M. Riesz's proofs of the theorem about conjugate functions, or in our own proof of the theorems referred to on p. 306, note †. See Hausdorff (**4**), M. Riesz (**7**), Hardy and Littlewood (**1**).

§ M. Riesz (**7**); see also Hardy, Littlewood, and Pólya (**3**), 214–19, Theorem 295).

‖ Which we stated without proof in (**2**).

Thus f_n is the 'Fourier polynomial' of f of degree n. We use a similar notation for other functions.

We denote by $F^{(n)}$ any integrable function such that $f_n \prec (F^{(n)})_n$, i.e. such that

$$|c_m| \leqslant C_m \qquad (|m| \leqslant n).$$

Thus $F^{(n)}$ depends to some extent upon f_n, but has an arbitrary 'tail', in which the coefficients may be complex. When $n \to \infty$, $F^{(n)}$ becomes a majorant of f. We define $\lambda_q(n)$ by

$$\lambda_q(n) = \max \frac{J_q(f_n)}{J_q(F^{(n)})}. \tag{2.1.1}$$

Here 'max' means 'upper bound', and the upper bound is taken over all f_n and all $F^{(n)}$ associated with each f_n. We shall see in a moment that this upper bound exists.

Next, let g be any function of L^p, G_n any polynomial of degree n which majorizes g_n, and

$$\mu_p(g, n) = \min \frac{J_p(G_n)}{J_p(g)}$$

(the lower bound for all such G_n). This lower bound is attained for one or many G_n, which we call G_n^*; and we can select one G_n^* so as to be uniquely defined by g.† We now define $\mu_p(n)$ by

$$\mu_p(n) = \max \mu_p(g, n) = \max \frac{J_p(G_n^*)}{J_p(g)}, \tag{2.1.2}$$

the upper bound being taken over all g and the uniquely corresponding G_n^*. In this case also we shall see that the upper bound exists.

2.2. Lemma 1. *The bounds $\lambda_q(n)$, $\mu_p(n)$ exist.*

(1) Given f_n, $F^{(n)}$, we have‡

$$|c_m| = |c_m(f_n)| \leqslant c_m(F^{(n)}) \leqslant J_1(F^{(n)}) \leqslant J_q(F^{(n)}) \qquad (|m| \leqslant n),$$

$$|f_n| \leqslant (2n+1)\max_{|m| \leqslant n} |c_m| \leqslant (2n+1) J_q(F^{(n)}),$$

$$\frac{J_q(f_n)}{J_q(F^{(n)})} \leqslant \frac{\max |f_n|}{J_q(F^{(n)})} \leqslant 2n+1.$$

Hence $\lambda_q(n) \leqslant 2n+1$.

† The existence of one G_n^* at any rate is shown by the 'Bolzano-Weierstrass' argument, applied to the space of $2(2n+1)$ real coordinates defined by the real and imaginary parts of the $2n+1$ coefficients.

If there are many G_n^*, let $u+iv$ be the central coefficient of a G_n^*, and select those for which u has its maximum value. If this is not enough to define a unique polynomial, apply the same process to v, and then, if necessary, to other coefficients.

‡ We use $c_m(h)$ generally for the mth Fourier coefficient of h. Without an explicit argument, c_m means $c_m(f)$.

(2) Given g, let G_n be the *exact* majorant of g_n, i.e.

$$G_n = \sum_{-n}^{n} |c_m(g)| e^{mi\theta}.$$

By the definition of G_n^*,

$$J_p(G_n^*) \leqslant J_p(G_n).$$

Also
$$c_m(G_n) = |c_m(g)| \leqslant J_1(g) \leqslant J_p(g),$$
$$J_p(G_n) \leqslant \max|G_n| \leqslant (2n+1)\max c_m(G_n) \leqslant (2n+1)J_p(g),$$

and so
$$\frac{J_p(G_n^*)}{J_p(g)} \leqslant \frac{J_p(G_n)}{J_p(g)} \leqslant 2n+1.$$

Thus both $\lambda_q(n)$ and $\mu_p(n)$ exist (and do not exceed $2n+1$).

2.3. In what follows we write generally

$$\chi_r(\zeta) = |\zeta|^{r-1}\operatorname{sgn}\bar{\zeta},$$

where the bar denotes the conjugate. Thus $\chi_r(\zeta) = \rho^{r-1}e^{-i\alpha}$ when $\zeta = \rho e^{i\alpha}$.

LEMMA 2. *If h is a trigonometrical polynomial, and t and γ are real, then*

$$\left(\frac{d}{dt}\frac{1}{2\pi}\int_{-\pi}^{\pi} |h+te^{i\gamma+mi\theta}|^r\,d\theta\right)_{t=0} = r\mathbf{R}\{e^{i\gamma}c_{-m}[\chi_r(h)]\}. \qquad (2.3.1)$$

We may differentiate under the integral sign. Also, if $h = \rho e^{i\alpha}$, we have

$$|h_t| = |h+te^{i\gamma+mi\theta}| = \sqrt{\{\rho^2+t^2+2\rho t\cos(m\theta+\gamma-\alpha)\}},$$
$$\frac{d}{dt}|h_t|^r = r|h_t|^{r-2}\{t+\rho\cos(m\theta+\gamma-\alpha)\},$$

$$\frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|h_t|^r\,d\theta\right)_{t=0} = \frac{r}{2\pi}\int_{-\pi}^{\pi}|h|^{r-1}\cos(m\theta+\gamma-\alpha)\,d\theta$$
$$= \frac{r}{2\pi}\mathbf{R}\left(\frac{e^{i\gamma}}{2\pi}\int_{-\pi}^{\pi}|h|^{r-1}\operatorname{sgn}\bar{h}\,e^{mi\theta}\,d\theta\right),$$

the formula required.

2.4. Our next lemma is decidedly more difficult, and its proof contains the kernel of the proof of Theorem 1.

LEMMA 3. *For every q and n*

$$\lambda_q(n) = \mu_{q'}(n). \qquad (2.4.1)$$

(1) We show first that
$$\lambda_q \leqslant \mu_{q'}; \qquad (2.4.2)$$

this is the easier half of the proof.

We start from an f, an f_n, not null, and an associated $F^{(n)}$, take

$$g = \chi_q(f_n) = |f_n|^{q-1}\operatorname{sgn} \bar{f}_n,$$

and denote by G_n^* the function associated with g in the manner of § 2.1. We write

$$k_m = c_m(g), \qquad K_m^* = c_m(G_n^*).$$

Then

$$J_q^q(f_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n g\, d\theta = \sum_{-n}^{n} c_m k_{-m}$$

$$\leqslant \sum_{-n}^{n} |c_m||k_{-m}| \leqslant \sum_{-n}^{n} C_m K_{-m}^*, \tag{2.4.3}$$

since $|c_m| \leqslant C_m$ and $|k_{-m}| \leqslant K_{-m}^*$, by the definitions of $F^{(n)}$ and G_n^* respectively. But

$$\sum_{-n}^{n} C_m K_{-m}^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^{(n)} G_n^*\, d\theta \leqslant J_q(F^{(n)}) J_{q'}(G_n^*). \tag{2.4.4}$$

Hence, by the definition of $\mu_{q'}$,

$$J_q^q(f_n) \leqslant J_q(F^{(n)}) J_{q'}(G_n^*) \leqslant J_q(F^{(n)}) . \mu_{q'} J_{q'}(g) = \mu_{q'} J_q(F^{(n)}) J_q^{q-1}(f_n),$$

$$\frac{J_q(f_n)}{J_q(F^{(n)})} \leqslant \mu_{q'}, \tag{2.4.5}$$

which proves (2.4.2).

(2) Secondly, we prove that

$$\mu_{q'} \leqslant \lambda_q. \tag{2.4.6}$$

Suppose that g is L^p, G_n^* (not null) is the function associated with g, and

$$k_m = c_m(g), \qquad K_m^* = c_m(G_n^*). \tag{2.4.7}$$

We write† $$H^{(n)} = \chi_p(G_n^*) = |G_n^*|^{p-1}\operatorname{sgn} \bar{G}_n^*, \tag{2.4.8}$$

$$L_m = c_m(H^{(n)}),$$

and we begin by proving some properties of the L_m.

(i) It follows from (2.4.8) and (2.4.7) that‡

$$H^{(n)} = |G_n^*|^{p-2}\bar{G}_n^* = |G_n^*|^{p-2} \sum_{-n}^{n} K_m^* e^{-mi\theta} \tag{2.4.9}$$

(since K_m^* is real). Since

$$|G_n^*(-\theta)| = |\overline{G_n^*(\theta)}| = |G_n^*(\theta)|,$$

† It will be shown later that $H^{(n)}$ is related to an h_n as $F^{(n)}$ is to f_n in § 2.1: the notation anticipates this.

‡ Unless $G_n^* = 0$, which can happen at most for a finite number of values of θ.

the first factor on the right of (2.4.9) is real and even, and has, therefore real Fourier coefficients. The second factor also has real coefficients, and so therefore has $H^{(n)}$. Hence

(*a*) *L_m is real for all m.*

(ii) If
$$K_m^* > |k_m|, \tag{2.4.10}$$
then, by the minimal property of G_n^*,
$$J_p^p(G_n^* + te^{mi\theta}) \geqslant J_p^p(G_n^*) \tag{2.4.11}$$
for small real t of either sign. Hence
$$\left\{\frac{d}{dt} J_p^p(G_n^* + te^{mi\theta})\right\}_{t=0} = 0;$$
and
$$L_{-m} = \mathbf{R}(L_{-m}) = \mathbf{R}\{c_{-m}(\chi_p(G_n^*))\} = 0,$$
by Lemma 2 (with $\gamma = 0$). Thus

(*b*) *$L_{-m} = 0$ if $K_m^* > |k_m|$.*

(iii) This argument fails when $K_m^* = |k_m|$, since $K_m = c_m(G)$ has to satisfy $K_m \geqslant |k_m|$, and t cannot be negative. But (2.4.11) is still true for *positive* t (and $|m| \leqslant n$), and so $L_{-m} \geqslant 0$. Hence

(*c*) *$L_{-m} \geqslant 0$ for $|m| \leqslant n$.*

Hence
$$(H^{(n)})_n = \sum_{-n}^{n} L_m e^{mi\theta}$$
has non-negative coefficients. And if we write
$$l_m = L_m \operatorname{sgn} \bar{k}_{-m} \quad (|m| \leqslant n), \qquad l_m = 0 \quad (|m| > n), \tag{2.4.12}$$
then $(H^{(n)})_n$ is a majorant of
$$h_n = \sum_{-n}^{n} l_m e^{mi\theta},$$
and

(*d*) *$H^{(n)}$ is of the type $H^{(n)}$ of* § 2.1, *i.e. is related to h_n as $F^{(n)}$ is to f_n in* § 2.1.

Now
$$J_p^p(G_n^*) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_n^* H^{(n)}\, d\theta = \sum_{-n}^{n} K_m^* L_{-m} = \sum_{-n}^{n} |k_m| L_{-m},$$
on account of (*b*). This, by (2.4.12), is
$$\sum_{-n}^{n} k_m l_{-m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g h_n\, d\theta \leqslant J_p(g) J_{p'}(h_n).$$

Combining these results, and using the definition of $\lambda_{p'}$, we obtain

$$J_p^p(G_n^*) \leqslant J_p(g).\lambda_{p'}J_{p'}(H^{(n)}) = \lambda_{p'}J_p(g)J_p^{p-1}(G_n^*),$$

$$\frac{J_p(G_n^*)}{J_p(g)} \leqslant \lambda_{p'},$$

and therefore (2.4.6). This completes the proof of Lemma 3.

2.5. LEMMA 4. *For every r, f, and n,*

$$J_r(f_n) \leqslant A_r J_r(f). \tag{2.5.1}$$

This is one of M. Riesz's theorems.†

LEMMA 5. *If* (1.4.1) *is true for a particular q, all f, and all majorants F of f, then*

$$J_q(f_n) \leqslant A_q J_q(F^{(n)}) \tag{2.5.2}$$

for every f_n and every $F^{(n)}$ associated with f_n as in § 2.1. *In particular* (2.5.2) *is true when $q = 2k$.*

Since $F_n^{(n)}$ is a majorant of f_n,

$$J_q(f_n) \leqslant A_q J_q(F_n^{(n)}) \leqslant A_q J_q(F^{(n)}),$$

by Lemma 4.

2.6. We can now prove Theorem 1. Since (2.5.2) is true for all f_n and $F^{(n)}$,

$$\lambda_q(n) \leqslant A_q,$$

and therefore, by Lemma 3,

$$\mu_{q'}(n) \leqslant A_q.$$

If then $p = q'$, $q = p'$, we have

$$J_p(G_n^*) \leqslant A_{p'}J_p(g)$$

for every n and every associated pair g and G_n^*. There is therefore‡ a sequence (n_ν) such that $G_{n_\nu}^*$ converges weakly to a G^* for which

$$J_p(G^*) \leqslant A_{p'}J_p(g).$$

Finally

$$c_m(G_{n_\nu}^*) \geqslant |c_m(g)|$$

if $n_\nu \geqslant m$, so that

$$c_m(G^*) = \lim_{\nu\to\infty} c_m(G_{n_\nu}^*) \geqslant |c_m(g)|$$

for all m, and G^* is a majorant of g.

† M. Riesz (8), Theorem 5.

‡ F. Riesz (5), § 7.

3. An appendix

3.1. We have now completed the substance of the paper, but we add a few minor observations. These do not take us far, but, since they may have some bearing on the unsolved problem, are probably worth recording. It is inevitable in the circumstances that they should seem rather disjointed.

We require

LEMMA 6. *If f is L^r,*

$$g = \chi_r(f) = |f|^{r-1}\operatorname{sgn}\bar{f},$$

and

$$k_{-m} = c_{-m}(g) = 0$$

for a particular m, then $J_r(f)$ is increased strictly by any alteration of c_m.

Let

$$f_1 = f + ce^{mi\theta}$$

where $c \neq 0$. If $J_r(f) = 0$, f is null, f_1 not null, and $J_r(f) < J_r(f_1)$. Otherwise we have

$$J_r^r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} fg\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1 g\,d\theta\dagger$$

$$\leqslant J_r(f_1)J_r(g) = J_r(f_1)J_r^{r-1}(f),$$

and so $J_r(f) \leqslant J_r(f_1)$. Equality demands

$$|f_1|^r \equiv |g|^{r'} \equiv |f|^r$$

and also

$$\operatorname{sgn} f_1 \equiv \operatorname{sgn}\bar{g} \equiv \operatorname{sgn} f;$$

so $f_1 \equiv f$, which is false.

3.2. THEOREM 2. *If*

$$|c_m| \leqslant C_m \quad (|m| \leqslant n)$$

and, among all polynomials f_n which satisfy these conditions, f_n^* gives a maximum for $J_r(f_n)$, then

$$|c_m^*| = C_m \quad (|m| \leqslant n).$$

If

$$g = \chi_r(f_n), \qquad k_m = c_m(g),$$

and

$$k_{-m} = 0, \tag{3.2.1}$$

then, by Lemma 6, any change in c_m increases $J_r(f_n)$ strictly. Hence f_n^* cannot give the maximum unless $C_m = 0$, when there is nothing to prove.

If $|c_m^*| < C_m$ then

$$J_r(f_n^* + te^{i\gamma}e^{mi\theta}) \leqslant J_r(f_n^*)$$

† Since $k_{-m} = 0$.

for all real γ and small positive t. Hence, by Lemma 2,

$$\mathbf{R}(e^{i\gamma}k_{-m}) = 0$$

for all real γ. This implies $k_{-m} = 0$, which is impossible because $C_m \neq 0$.

3.3. From Theorem 2 we can deduce another theorem bearing on our main problem.

It is easy to see that there are 'maximal pairs' f_n, $F^{(n)}$ for which

$$J_q(f_n) = \lambda_q(n) J_q(F^{(n)}).\dagger$$

For such a pair f_n, $F^{(n)}$, all of (2.4.3)–(2.4.5) become equalities. From this it follows that g, G_n^* are a maximal pair in the conjugate problem, and that

$$G_n^* \equiv \lambda_q \chi_q(F^{(n)}) = \lambda_q |F^{(n)}|^{q-1} \operatorname{sgn} \overline{F}^{(n)}.$$

Also $$|c_m||k_{-m}| = C_m K_{-m}^*,$$

and so $$|c_m| = C_m \qquad (|m| \leqslant n)$$

unless $$k_{-m} = 0.$$

Theorem 2, however, enables us to say more.

THEOREM 3. *If f_n and $F^{(n)}$ are a maximal pair, then $F_n^{(n)}$ is the exact majorant of f_n.*

For, if this were not so, we could leave $F^{(n)}$ fixed, and increase $J_q(f_n)/J_q(F^{(n)})$ by changing f_n.

Finally we prove

THEOREM 4. *If* $$0 \leqslant c_m \leqslant C_m,$$

and, among all polynomials f_n which satisfy these conditions, f_n^* gives a maximum for $J_r(f_n)$, then every c_m^* is 0 or C_m.

In the first place, as in § 2.4 (2) (i),

$$g = \chi_r(f) = |f|^{r-1} \operatorname{sgn} \bar{f}$$

has real Fourier coefficients.

Next, as in the proof of Theorem 2,

$$k_{-m} = c_{-m}(g) = 0,$$

† We select a maximizing sequence $f_n^{(N)}$, $F^{(n,N)}$. Then $f_n^{(N)}$ has a limit function f_n, when $N \to \infty$ appropriately. Also $F^{(n,N)}$ has a weak limit $F^{(n)}$ for a subsequence of N, and

$$J_q(F^{(n)}) \leqslant \underline{\lim}\, J_q(F^{n,N}).$$

Hence $$J_q(f_n) \geqslant \lambda_q(n) J_q(F^{(n)});$$

and the opposite inequality is a consequence of the definition of $\lambda_q(n)$.

if $0 < c_m < C_m$; and then, by Lemma 6, any change in c_m will increase $J_r(f_n)$. Hence, for the maximal polynomial, c_m is 0 or C_m.

There are other proofs of Theorems 2–4, but we have chosen those connected most closely with the analysis of § 2.

REFERENCES

1. G. H. Hardy and J. E. Littlewood, 'Some new properties of Fourier constants': *Math. Annalen*, 97 (1926), 159–209.
2. ——, 'Notes on the theory of series (xiii): Some new properties of Fourier constants', *J. of London Math. Soc.*, 6 (1931), 3–9.
3. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge, 1934).
4. F. Hausdorff, 'Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen': *Math. Zeitschrift*, 16 (1923), 163–9.
5. F. Riesz, 'Untersuchungen über Systeme integrierbarer Funktionen': *Math. Annalen*, 69 (1910), 449–97.
6. ——, 'Über eine Verallgemeinerung der Parsevalsche Formeln': *Math. Zeitschrift*, 18 (1923), 117–24.
7. M. Riesz, 'Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires': *Acta Math.* 49 (1927), 465–97.
8. ——, 'Sur les fonctions conjuguées': *Math. Zeitschrift*, 27 (1928), 218–44.

A SUPPOSED PROPERTY OF WAVE-FRONTS

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It is sometimes stated* in text-books of physics that the wave-fronts in a progressive train of waves are the loci of points at which the oscillations are in the same phase at any instant. There is no obvious reason why these loci should be the envelopes of secondary wave-fronts which appear in the elementary applications of Huyghens's principle. Clearly the statement only concerns wave-trains in which the oscillations are simple-harmonic, but, even so, the property does not belong to every solution, of this type, of the equation of wave-motion

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

Consider, for example,

$$\phi = x\cos n(t-z/c)+y\sin n(t-z/c).$$

Here the 'waves' travel in the direction of the z-axis. The 'wave-fronts' are the planes normal to this, and these are not the loci of points of equal phase. If then equality of phase is a necessary property of wave-fronts, it must be for the physical reason, if it is a true one, that this is the manner in which waves are actually propagated.

Now the study of geometrical optics introduces us to systems of rays. Such a system of rays is always a congruence of normals to a system of parallel surfaces (only homogeneous isotropic media are under consideration). According to Huyghens's principle, the parallel surfaces are the wave-fronts. This is the geometrical basis of the calculations of the forms of wave-surfaces given by Bouasse,† who, however, points out carefully that the wave-surface he describes has no necessary connexion with undulatory theory. His wave-surface is, in fact, a surface of equal optical distance.

I have failed to find any suitable mathematical description of a progressive train of waves in three dimensions, and, for my own part, am unable to frame one which is free from ambiguity. This will not affect the negative theorem which I shall establish.

* e.g. R. W. Wood, *Physical Optics* (1919 edition), 9; A. Schuster and J. W. Nicholson, *Physical Optics* (1924), 42.

† H. Bouasse, *Optique géométrique supérieure* (1926), 25, 31–4.

THEOREM. *Solutions representing stationary oscillations are the only solutions of the equation of wave-motion such that, on each of a given system of parallel surfaces, the oscillations are in the same phase at the same instant.*

Suppose we are given a system of parallel surfaces. We select any one of them for purposes of reference, and call the normal distance of a point from this surface ν. On the surface $\nu = 0$, we take the lines of curvature as parametric curves $\lambda =$ constant, $\mu =$ constant. The two systems of developable surfaces generated by normals to the surface $\nu = 0$ at points on lines of curvature are then the surfaces $\lambda =$ constant, $\mu =$ constant; and λ, μ, ν constitute a set of orthogonal curvilinear coordinates.

Let the two fundamental forms* of the surface $\nu = 0$ be

$$E\,d\lambda^2 + G\,d\mu^2, \qquad D\,d\lambda^2 + D''\,d\mu^2.$$

It follows that the first fundamental form of the surface $\nu =$ constant is

$$\frac{(E - D\nu)^2}{E}\,d\lambda^2 + \frac{(G - D''\nu)^2}{G}\,d\mu^2,$$

so that, for any displacement ds,

$$ds^2 = \frac{(E - D\nu)^2}{E}\,d\lambda^2 + \frac{(G - D''\nu)^2}{G}\,d\mu^2 + d\nu^2.$$

It is convenient to write $E = A^2$, $D = AB$, $G = C^2$, $D'' = CD$. [The change of meaning of D will cause no confusion.] We now have

$$ds^2 = (A - B\nu)^2\,d\lambda^2 + (C - D\nu)^2\,d\mu^2 + d\nu^2,$$

so that, in a usual notation,

$$\frac{1}{h_1} = A - B\nu, \qquad \frac{1}{h_2} = C - D\nu, \qquad \frac{1}{h_3} = 1, \tag{1}$$

and the equation of wave-motion takes the form

$$\nabla^2\phi \equiv h_1 h_2 \left\{ \frac{\partial}{\partial\lambda}\left(\frac{h_1}{h_2}\frac{\partial\phi}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(\frac{h_2}{h_1}\frac{\partial\phi}{\partial\mu}\right) + \frac{\partial}{\partial\nu}\left(\frac{1}{h_1 h_2}\frac{\partial\phi}{\partial\nu}\right) \right\} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}. \tag{2}$$

Suppose that a value of ϕ satisfying this equation is

$$U\cos nt - V\sin nt,$$

where U, V are functions of λ, μ, ν. If only one term is present, say $U\cos nt$, ϕ plainly represents stationary oscillations and the phase

* The notation is that of L. P. Eisenhart, *Differential Geometry* (1909). For the forms employed see pp. 122, 141, 142, 178.

of the oscillations at any instant is the same throughout any space in which U is free from singularities. Both terms must, therefore, be present for our purpose. By writing $t = t' - \pi/2n$, we see that $U \sin nt + V \cos nt$ also satisfies the equation, and therefore We^{int}, where $W = U + iV$, satisfies it. Hence

$$\nabla^2 W = -\frac{n^2}{c^2} W.$$

We now write $W = Re^{i\theta}$, and, in order that the phase of the oscillations may be the same at all points of any surface $\nu =$ constant, we must make θ a function of ν only.

Now

$$\frac{\partial}{\partial \nu}\left\{\frac{1}{h_1 h_2}\frac{\partial}{\partial \nu}(Re^{i\theta})\right\} = e^{i\theta}\left\{\frac{\partial}{\partial \nu}\left(\frac{1}{h_1 h_2}\frac{\partial R}{\partial \nu}\right) + \frac{2i}{h_1 h_2}\frac{\partial R}{\partial \nu}\frac{d\theta}{d\nu} + \right.$$
$$\left. + iR\frac{\partial}{\partial \nu}\left(\frac{1}{h_1 h_2}\right)\frac{d\theta}{d\nu} + \frac{iR}{h_1 h_2}\frac{d^2\theta}{d\nu^2} - \frac{R}{h_1 h_2}\left(\frac{d\theta}{d\nu}\right)^2\right\},$$

so that

$$\nabla^2 R + i\left\{2\frac{\partial R}{\partial \nu}\frac{d\theta}{d\nu} + Rh_1 h_2 \frac{\partial}{\partial \nu}\left(\frac{1}{h_1 h_2}\right)\frac{d\theta}{d\nu} + R\frac{d^2\theta}{d\nu^2}\right\} - R\left(\frac{d\theta}{d\nu}\right)^2 = -\frac{n^2}{c^2} R.$$

Hence
$$\nabla^2 R = R\left\{\left(\frac{d\theta}{d\nu}\right)^2 - \frac{n^2}{c^2}\right\}, \tag{3}$$

$$2\frac{\partial R}{\partial \nu}\frac{d\theta}{d\nu} + Rh_1 h_2 \frac{\partial}{\partial \nu}\left(\frac{1}{h_1 h_2}\right)\frac{d\theta}{d\nu} + R\frac{d^2\theta}{d\nu^2} = 0. \tag{4}$$

Now $d\theta/d\nu$ cannot vanish. If it did, θ would be a constant, say α, and the value of ϕ would be $R\cos(nt+\alpha)$, which represents a stationary oscillation. Hence we may write (4) as

$$\frac{2}{R}\frac{\partial R}{\partial \nu} + h_1 h_2 \frac{\partial}{\partial \nu}\left(\frac{1}{h_1 h_2}\right) + \frac{d^2\theta}{d\nu^2}\Big/\frac{d\theta}{d\nu} = 0,$$

i.e.
$$\frac{R^2}{h_1 h_2}\frac{d\theta}{d\nu} = \chi^2, \tag{5}$$

where χ is a function of λ, μ. It is convenient to write $d\theta/d\nu = 1/N^2$, and we must now substitute in (3) the precise form of the operator ∇^2 given by (1), (2),

i.e.

$$\frac{\partial}{\partial \lambda}\left(\frac{C-D\nu}{A-B\nu}\frac{\partial R}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{A-B\nu}{C-D\nu}\frac{\partial R}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left\{(A-B\nu)(C-D\nu)\frac{\partial R}{\partial \nu}\right\}$$
$$= (A-B\nu)(C-D\nu)R\left(\frac{1}{N^4} - \frac{n^2}{c^2}\right) \tag{6}$$

where, from (5), $$R = \frac{N\chi}{(A-B\nu)^{\frac{1}{4}}(C-D\nu)^{\frac{1}{4}}}.$$

When the substitution of this value of R is made in (6), the resulting equation can be written as

$$(A-B\nu)^4(C-D\nu)^4\left(\frac{1}{N^4}-\frac{n^2}{c^2}-\frac{1}{N}\frac{d^2N}{d\nu^2}\right)-\frac{1}{4}(AD-BC)^2(A-B\nu)^2(C-D\nu)^2$$

$$= (C-D\nu)^4(A-B\nu)^2\frac{1}{\chi}\frac{\partial^2\chi}{\partial\lambda^2}-2\left(\frac{\partial A}{\partial\lambda}-\nu\frac{\partial B}{\partial\lambda}\right)(C-D\nu)^4(A-B\nu)\frac{1}{\chi}\frac{\partial\chi}{\partial\lambda}-$$

$$-\frac{1}{2}\left(\frac{\partial^2 A}{\partial\lambda^2}-\nu\frac{\partial^2 B}{\partial\lambda^2}\right)(C-D\nu)^4(A-B\nu)+\frac{5}{4}\left(\frac{\partial A}{\partial\lambda}-\nu\frac{\partial B}{\partial\lambda}\right)^2(C-D\nu)^4+$$

$$+\frac{1}{2}\left(\frac{\partial A}{\partial\lambda}-\nu\frac{\partial B}{\partial\lambda}\right)\left(\frac{\partial C}{\partial\lambda}-\nu\frac{\partial B}{\partial\lambda}\right)(C-D\nu)^3(A-B\nu)-$$

$$-\frac{1}{2}\left(\frac{\partial^2 C}{\partial\lambda^2}-\nu\frac{\partial^2 D}{\partial\lambda^2}\right)(A-B\nu)^2(C-D\nu)^3+$$

$$+\frac{1}{4}\left(\frac{\partial C}{\partial\lambda}-\nu\frac{\partial D}{\partial\lambda}\right)^2(A-B\nu)^2(C-D\nu)^2+$$

$+$ similar terms in which the pairs of symbols λ, μ; A, C; B, D are interchanged.

Now suppose values are assigned arbitrarily to λ, μ. For all points on the surface $A = B\nu$,

$$\frac{1}{A}\frac{\partial A}{\partial\lambda} = \frac{1}{B}\frac{\partial B}{\partial\lambda}, \quad \text{or} \quad AD = BC.$$

As the values of λ, μ are assigned arbitrarily, one of these alternatives must be true universally.

The former of these gives $B = AM$, and we may find, in a similar manner that $C = DL$, where L, M are functions of λ, μ only. Thus, in the notation of Eisenhart, either

$$\frac{D}{E} = \frac{D''}{E''} \quad \text{or} \quad D = EM, \quad D'' = GL.$$

In general, neither alternative is true, and the theorem is established. Consequently wave-fronts in general cannot have the property stated.

Consider now a special solution of the equation of wave-motion,

$$\phi = \frac{\partial^2}{\partial z^2}\left\{\frac{\cos n(t-r/c)}{r}\right\}$$

$$= \frac{n}{c}\left(\frac{1}{r^2}-\frac{3z^2}{r^4}\right)\sin n(t-r/c)-\left(\frac{1}{r^3}-\frac{3z^2}{r^5}-\frac{n^2z^2}{c^2r^3}\right)\cos n(t-r/c).$$

where, as usual, r denotes the distance of the point (x, y, z) from the origin of coordinates. This plainly represents a spherical wave-system. The irregularity of phase, which has been shown to exist in general, depends here on the variation of

$$\tan^{-1}\frac{r(r^2-3z^2)n/c}{r^2-3z^2-r^2z^2n^2/c^2}, \quad \text{i.e.} \quad \tan^{-1}\frac{2\pi\lambda r(r^2-3z^2)}{\lambda^2(r^2-3z^2)-4\pi^2r^2z^2},$$

where λ is the length of plane waves of the same frequency. When r is large in comparison with λ, the irregularity is negligible. This suggests that it can only be of importance when one at least of the radii of curvature of the wave-front is of the same order as the wave-length. It seems highly improbable that any theorem of physical optics is affected by what has been proved.

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